The O(N) Vector Model

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Review

- In lecture 22, we discussed the QFT of a complex scalar field ϕ
- Separating ϕ into real and imaginary parts $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$ we found

$$Z = \int \mathcal{D}\phi_1 \mathcal{D}\phi_2 e^{-S_E[\phi_1,\phi_2]}\,, \tag{26.1}$$

  for the QFT partition function
- In this lecture, we will consider an N-component scalar field

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \ldots \\ \phi_N \end{pmatrix} \tag{26.2}$$

  as a generalization of the complex scalar field

The Action for the Complex Scalar Field

- For a single complex scalar field, we had in (22.4) the Euclidean action

$$S_E = \int_s \left[\partial_a \phi \partial_a \phi^* + m^2 \phi \phi^* + 4\lambda \left(\phi \phi^*\right)^2\right] . \qquad (26.3)$$

- We found (26.3) has an additional symmetry: it is invariant under the transformation

$$\phi(x) \to e^{i\alpha} \phi(x) , \quad \left(\phi^*(x) \to e^{-i\alpha} \phi^*(x)\right) , \qquad (26.4)$$

with arbitrary (but constant) α

- This is called a U(1) transformation, for a unitary 1x1 matrix

The Action for the Complex Scalar Field

- In components $\phi = \frac{1}{\sqrt{2}}\left(\phi_1 + i\phi_2\right)$ we had in (22.7):

$$S_E = \int_x \left[\frac{1}{2}\partial_a\phi_1\partial_a\phi_1 + \frac{1}{2}\partial_a\phi_2\partial_a\phi_2 + \frac{m^2}{2}\phi_1^2 + \frac{m^2}{2}\phi_2^2 + \lambda\left(\phi_1^2+\phi_2^2\right)^2\right]. \tag{26.5}$$

- The U(1) symmetry (26.4) now becomes

$$\begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}, \tag{26.6}$$

  where we can think of ϕ_1, ϕ_2 as the coordinates of a vector in a plane
- With this interpretation, (26.6) is the rotation of the vector in the plane, also called an SO(2) transformation, for a special (unit determinant) orthogonal 2x2 matrix

The O(N) Vector Model

- Let us now consider a generalization of the 2-component vector $\begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}$ to a vector with N scalar field components

$$\vec{\phi} = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \\ \ldots \\ \phi_N(x) \end{pmatrix} . \tag{26.7}$$

- By analogy with the complex scalar field, we expect the Euclidean action to be invariant under an SO(N) symmetry (in addition to the usual Lorentz invariance)
- In 3+1 dimensions, one such action that generalizes (26.5) is

$$S_E = \int_x \left[\frac{1}{2}\partial_a \vec{\phi} \cdot \partial_a \vec{\phi} + \frac{m^2}{2} \vec{\phi} \cdot \vec{\phi} + \frac{2\lambda}{N} \left(\vec{\phi} \cdot \vec{\phi} \right)^2 \right] . \tag{26.8}$$

The O(N) Vector Model

- The QFT that arises from the classical action (26.8) is called the O(N)-vector model
- The partition function for the O(N) vector model is given by

$$Z = \int \mathcal{D}\vec{\phi} e^{-S_E} \,. \tag{26.9}$$

- For N=2, the O(N) vector model partition function is identical to Z for the complex scalar field
- For N=1, the O(N) vector model partition function is identical to Z for the real scalar field, with double the coupling constant

Solving the O(N) Vector Model

- We can treat the interaction term $\lambda\left(\vec{\phi}\cdot\vec{\phi}\right)^2$ in perturbation theory just like for the real scalar field
- However, we have a huge advantage in the O(N) model over a real scalar field QFT: we can solve the theory exactly in the limit $N \gg 1$
- This is a rare case where one does *not* need perturbation theory to study a QFT
- I'll be covering the basics here, but advanced students may find the following reference useful: https://arxiv.org/pdf/1905.09290.pdf

Solving the O(N) Vector Model

- To solve the O(N) model in the large N limit, first insert unity in the path integral for the partition function:

$$Z = \int \mathcal{D}\vec{\phi} e^{-S_E} \times 1 \,. \tag{26.10}$$

- Next, write unity as a (path-) integral over a δ function

$$1 = \int \mathcal{D}\sigma \delta \left(\sigma - \frac{\vec{\phi}\cdot\vec{\phi}}{N}\right) \tag{26.11}$$

- Use the δ function to replace the quartic term in the action by σ^2

Solving the O(N) Vector Model

- Next, write the δ function in integral representation as

$$\delta\left(\sigma-\frac{\vec{\phi}\cdot\vec{\phi}}{N}\right)=\int\mathcal{D}\zeta e^{i\int_x\zeta\left(\sigma-\frac{\vec{\phi}\cdot\vec{\phi}}{N}\right)} \tag{26.12}$$

- We get for Z:

$$Z=\int\mathcal{D}\vec{\phi}\mathcal{D}\sigma\mathcal{D}\zeta e^{-\frac{1}{2}\int_x\vec{\phi}\left[-\partial_a^2+m^2+\frac{2i\zeta}{N}\right]\vec{\phi}-2\lambda N\int_x\sigma^2+i\int_x\zeta\sigma}\,. \tag{26.13}$$

- The path integral over σ is Gaussian, we can integrate out σ to find

$$Z=\int\mathcal{D}\vec{\phi}\mathcal{D}\zeta e^{-\frac{1}{2}\int_x\vec{\phi}\left[-\partial_a^2+m^2+\frac{2i\zeta}{N}\right]\vec{\phi}-\frac{1}{8\lambda N}\int_x\zeta^2}\,. \tag{26.14}$$

Solving the O(N) Vector Model

- Letting $\zeta \to N \times \zeta$ gives

$$Z = \int \mathcal{D}\vec{\phi}\mathcal{D}\zeta e^{-\frac{1}{2}\int_x \vec{\phi}\left[-\partial_a^2+m^2+2i\zeta\right]\vec{\phi}-\frac{N}{8\lambda}\int_x \zeta^2}\,. \tag{26.15}$$

- Separating ζ now into a "mean-field" part and fluctuations $\zeta(x) = \bar{\zeta} + \zeta'(x)$ as in lecture 19 gives

$$Z = \int d\bar{\zeta} \int \mathcal{D}\vec{\phi}\mathcal{D}\zeta' e^{-\frac{1}{2}\int_x \vec{\phi}\left[-\partial_a^2+m^2+2i(\bar{\zeta}+\zeta')\right]\vec{\phi}-\frac{N\beta V}{8\lambda}\bar{\zeta}^2-\frac{N}{8\lambda}\int_x \zeta'^2}\,. \tag{26.16}$$

- So far everything is exact for all N
- Now let's consider the limit $N \to \infty$

Solving the O(N) Vector Model

- For $N \to \infty$, the path integral over ζ' gives a contribution of order $e^{\ln N}$ to Z
- But the mean-field term is $e^N \gg e^{\ln N}$ in the large N limit
- So in the large N limit, neglecting the path integral over ζ' becomes exact and we get

$$\lim_{N\gg 1} Z = \int d\bar{\zeta} \int \mathcal{D}\vec{\phi} e^{-\frac{1}{2}\int_x \vec{\phi}\left[-\partial_a^2+m^2+2i\bar{\zeta}\right]\vec{\phi}-\frac{N\beta V}{8\lambda}\bar{\zeta}^2} \,. \qquad (26.17)$$

- The remaining path integral over the O(N) vector field $\vec{\phi}$ is Gaussian, and is given by N-copies of the real scalar field partition function,

$$\lim_{N\gg 1} Z = \int d\bar{\zeta} e^{N \ln Z_{\rm free}(T,\sqrt{m^2+2i\bar{\zeta}})-\frac{N\beta V}{8\lambda}\bar{\zeta}^2} \,, \qquad (26.18)$$

where the "mass" of the real scalar field is $\sqrt{m^2+2i\bar{\zeta}}$

Solving the O(N) Vector Model

- The remaining integral over $\bar{\zeta}$ can be evaluated from the saddle point of the integral
- For $N \to \infty$, the saddle point approximation is not an approximation, but becomes exact
- Denoting the position of the saddle as $\bar{\zeta} = \tilde{\zeta}$, we have

$$\lim_{N \gg 1} Z = e^{N \ln Z_{\rm free}(T, \sqrt{m^2 + 2i\tilde{\zeta}}) - \frac{N\beta V}{8\lambda}\tilde{\zeta}^2}\,. \qquad (26.19)$$

- Using the thermodynamic relation $p = \frac{\ln Z}{\beta V}$ this can be written as

$$\lim_{N \gg 1} Z = e^{N\beta V\left[p_{\rm free}(T, \sqrt{m^2 + 2i\tilde{\zeta}}) - \frac{\tilde{\zeta}^2}{8\lambda}\right]}\,. \qquad (26.20)$$

or

$$p(T, m, \lambda) = N\left[p_{\rm free}(T, \sqrt{m^2 + 2i\tilde{\zeta}}) - \frac{\tilde{\zeta}^2}{8\lambda}\right]\,. \qquad (26.21)$$

Solving the O(N) Vector Model

- The **exact result** for the QFT pressure of the O(N) model depends on the coupling *explicitly* as well a *implicitly* through the saddle point condition

$$\frac{\partial}{\partial\tilde{\zeta}} p_{\rm free}(T, \sqrt{m^2 + 2i\tilde{\zeta}}) - \frac{\tilde{\zeta}}{4\lambda} = 0\,. \tag{26.22}$$

- The free pressure for a single scalar field in 3+1 dimensions is divergent – we will discuss **nonperturbative** renormalization of the theory in the next lecture
- We will discuss how to evaluate the solution (26.21) in face of the condition (26.22) in the next lectures