# The O(N) Vector Model II

paul.romatschke@colorado.edu

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# Review

- In lecture 26, we solved the O(N) vector model in the large N limit
- We found

$$p(T,m,\lambda)=N\left[p_{\rm free}(T,\sqrt{m^2+2i\tilde{\zeta}})-\frac{\tilde{\zeta}^2}{8\lambda}\right]\,. \tag{27.1}$$

  for the pressure of the O(N) model, where $p_{\rm free}$ is the free real scalar field pressure
- Here $\tilde{\zeta}$ is the location of the saddle defined by the condition

$$\frac{\partial}{\partial\tilde{\zeta}}p_{\rm free}(T,\sqrt{m^2+2i\tilde{\zeta}})-\frac{\tilde{\zeta}}{4\lambda}=0\,. \tag{27.2}$$

- We will renormalize this result for $m=0$ in this lecture

# Regularization

- We have calculated the free scalar field pressure in lecture 11, finding in $\overline{\mathrm{MS}}$:

$$p_{\mathrm{free}}(T,m) = \frac{m^4}{64\pi^2}\left[\frac{1}{\varepsilon} + \ln\left(\frac{\bar{\mu}^2 e^{\frac{3}{2}}}{m^2}\right)\right] - J_B(T,m) \tag{27.3}$$

- While we want to set $m = 0$, the saddle point condition (27.2) involves $p_{\mathrm{free}}$ evaluated with the *effective mass* $\sqrt{m^2 + 2i\tilde{\zeta}}$
- Writing $x = i\tilde{\zeta}$ and setting $m = 0$, the saddle point condition (27.2) becomes

$$\frac{x}{8\pi^2}\left[\frac{1}{\varepsilon} + \frac{2\pi^2}{\lambda} + \ln\left(\frac{\bar{\mu}^2 e^1}{2x}\right)\right] - I_B(T,\sqrt{2x}) = 0 \tag{27.4}$$

# Renormalization of the O(N) model

- The physical result for the pressure of the O(N) model depends implicitly on the value of the saddle $x$
- The value of the saddle is determined non-perturbatively from (27.4)
- Perturbative renormalization is not an option for this result
- However, in this case we can perform a **non-perturbative** renormalization by simply putting

$$\frac{1}{2\pi^2\varepsilon}+\frac{1}{\lambda}=\frac{1}{\lambda_{\rm phys}}\,, \tag{27.5}$$

  with $\lambda_{\rm phys}$ the renormalized coupling constant.
- In this case, the saddle point condition is finite for all $\lambda_{\rm phys}$:

$$\frac{x}{8\pi^2}\left[\frac{2\pi^2}{\lambda_{\rm phys}}+\ln\left(\frac{\bar\mu^2 e^1}{2x}\right)\right]-I_B(T,\sqrt{2x})=0 \tag{27.6}$$

# Renormalization of the O(N) model

- Writing $x = i\tilde{\zeta}$, the $m = 0$ pressure (27.1) for the O(N) model is given by

$$p(T,\lambda) = N\left[\frac{x^2}{16\pi^2}\left(\frac{1}{\varepsilon} + \ln\left(\frac{\bar{\mu}^2 e^{\frac{3}{2}}}{2x}\right)\right) - J_B(T,\sqrt{2x}) + \frac{x^2}{8\lambda}\right] \tag{27.7}$$

- Using the non-perturbative renormalization condition (27.5), the divergence exactly cancels and we find

$$p(T,\bar{\mu}) = N\left[\frac{x^2}{16\pi^2}\ln\left(\frac{\bar{\mu}^2 e^{\frac{3}{2}}}{2x}\right) - J_B(T,\sqrt{2x}) + \frac{x^2}{8\lambda_{\rm phys}}\right] \tag{27.8}$$

# Renormalization of the O(N) model

- In the large N limit, the O(N) model is non-perturbatively renormalizable
- For $m = 0$ in dim.-reg, the theory only requires coupling-constant renormalization, and in particular *no cosmological constant counterterm*
- Renormalizing $\lambda$, the O(N) model pressure is automatically finite in dim.-reg, and can be evaluated non-perturbatively
- Let us now study the properties of this solution

# $\beta$-function of the O(N) model

- The pressure (27.8) is a physical quantity, and hence cannot depend on the arbitrary choice $\bar{\mu}$
- Just like in lecture 21, we thus must have $\lambda_{\rm phys}(\bar{\mu})$ dependent on $\bar{\mu}$ and

$$\bar{\mu}\frac{dp(T,\lambda_{\rm phys})}{d\bar{\mu}} = Nx^2\left[\frac{1}{8\pi^2}+\frac{d}{d\ln\bar{\mu}}\frac{1}{8\lambda_{\rm phys}}\right]=0 \qquad (27.9)$$

(Note that we used stationarity $\frac{dp}{dx}=0$ to simplify this expression)

- Clearly, we must have

$$\beta=\bar{\mu}\frac{\partial\lambda_{\rm phys}}{\partial\bar{\mu}}=\frac{\lambda_{\rm phys}^2}{\pi^2} \qquad (27.10)$$

which is the *exact* $\beta$-function in the large N limit for all $\lambda$

# $\beta$-function of the O(N) model

- The positive $\beta$-function for all $\lambda$ implies

$$\lambda_{\rm phys}(\bar{\mu}) = \frac{2\pi^2}{\ln \frac{\mu_0^2}{\bar{\mu}^2}}\,, \tag{27.11}$$

  which is the exact running coupling constant in the O(N) model
- Here $\mu_0$ is the Landau pole of the theory where $\lambda_{\rm phys}(\bar{\mu} = \mu_0) = \infty$.
- For small $\bar{\mu} \ll \mu_0$, the coupling constant is small, but it grows with growing energy scale, cf. lecture 21
- The O(N) model does not have a good continuum (high energy) limit $\bar{\mu} \to \infty$ since it becomes infinitely coupled at $\bar{\mu} = \mu_0$
- We can still treat it as an 'effective theory' valid for $\bar{\mu} \ll \mu_0$

# Cosmological constant of the O(N) model

- For $T = 0$, the pressure (27.8) corresponds to the cosmological constant of the theory
- Unlike other theories we discussed, the cosmological constant in the O(N) model is finite and can be calculated:

$$p(T=0,\lambda) = \frac{Nx^2}{16\pi^2}\left[\ln\left(\frac{\bar{\mu}^2 e^{\frac{3}{2}}}{2x}\right) + \frac{2\pi^2}{\lambda_{\rm phys}(\bar{\mu})}\right]. \tag{27.12}$$

- Plugging in the explicit coupling constant (27.11) this becomes

$$p(T=0,\lambda) = \frac{Nx^2}{16\pi^2}\ln\left(\frac{\mu_0^2 e^{\frac{3}{2}}}{2x}\right) \tag{27.13}$$

where $x$ is the solution to the saddle point condition (27.2):

$$\frac{x}{8\pi^2}\ln\left(\frac{\mu_0^2 e^1}{2x}\right) = 0 \tag{27.14}$$

# Cosmological constant of the O(N) model

- Using the saddle point condition, the cosmological constant becomes

$$p(T=0,\lambda)=\frac{Nx}{16\pi^2}\,. \tag{27.15}$$

- The saddle point condition has two solutions,

$$x=0\,,\quad x=\frac{\mu_0^2 e^1}{2}\,. \tag{27.16}$$

- The solution with $x\neq 0$ is proportional to the Landau pole $\mu_0$; at this high scale, the theory breaks down; we must therefore discard this solution as unreliable
- The only physical solution is $x=0$, so the cosmological constant in this case vanishes:

$$p(T=0,\lambda)=0\,. \tag{27.17}$$

# Cosmological constant of the O(N) model

- The result
$$p(T=0,\lambda)=0\,. \tag{27.18}$$
is very appealing, but potentially misleading
- Recall that in dimensional regularization, only logarithmic divergencies are registered
- By contrast, in cut-off regularization, the zero-temperature pressure would contain terms such as $\Lambda^4, \Lambda^2$, cf. lecture 11
- These would require additional counterterms in cut-off regularization
- **Dimensional regularization results for the cosmological constant need to be interpreted with great care!**