Graßmann Variables

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Anti-commuting numbers

- Normal numbers are commuting, e.g.

$$ab = ba \tag{29.1}$$

- However, one can also consider *anti-commuting* numbers, e.g.

$$ab = -ba\,. \tag{29.2}$$

- These anti-commuting numbers are called *Graßmann* (English spelling: Grassmann) numbers
- An important consequence of (29.2) is *nilpotence*, e.g.

$$a^2 = a^3 = a^4 = \ldots = 0\,. \tag{29.3}$$

Graßmann Numbers

- It is possible to represent Graßmann numbers as matrices
- For example, consider the two Graßmann numbers θ_1, θ_2 represented as

$$\theta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{29.4}$$

- It is straightforward to verify

$$\theta_1^2 = 0\,, \quad \theta_2^2 = 0\,, \quad \theta_1\theta_2 = -\theta_2\theta_1\,. \tag{29.5}$$

Graßmann integration

- As a consequence of nilpotence, Taylor expansions terminate after the first two terms:

$$f(\theta) = c_0 + c_1\theta + c_2\theta^2 + \ldots = c_0 + c_1\theta\,, \tag{29.6}$$

  with usual (commuting) coefficients c_0, c_1
- A surprising consequence is the unusual Graßmann integration rule
- Consider the Graßmann integral

$$\int d\theta f(\theta) = \int d\theta \left(c_0 + c_1\theta\right) = F(c_0, c_1)\,, \tag{29.7}$$

  where F is an arbitrary function of c_0, c_1

Graßmann integration

- Because the integral $\int d\theta f(\theta)$ is linear in c_0, c_1, we must have

$$F(c_0, c_1) = \alpha c_0 + \beta c_1 \,, \tag{29.8}$$

  with α, β ordinary c-numbers
- Furthermore, if we require symmetry under translations: $\theta \to \theta + \eta$

$$F(c_0, c_1) = \int d\theta \left(c_0 + c_1(\theta + \eta)\right) = F(c_0 + \eta c_1, c_1) \,, \tag{29.9}$$

- We find $\alpha = 0$, and arbitrary normalize $\beta = 1$ such that

$$\int d\theta \left(c_0 + c_1 \theta\right) = c_1 \tag{29.10}$$

Graßmann integration

- As a consequence, we find the following rules for Graßmann integration
$$\int d\theta\, 1 = 0\,, \quad \int d\theta\, \theta = 1\,. \tag{29.11}$$
- These rules are identical to *differentiation rules* for ordinary (commuting) variables:
$$\frac{\partial}{\partial x} 1 = 0\,, \quad \frac{\partial}{\partial x} x = 1\,. \tag{29.12}$$

Multiple Graßmann variables

- Let us now consider two different Graßmann variables: θ, η
- These fulfill

$$\{\theta, \eta\} = 0\,. \tag{29.13}$$

- Two-dimensional integrals are given by

$$\int d\eta d\theta f(\theta, \eta)\,, \tag{29.14}$$

where the integral measures also anti-commute, e.g.

$$\{d\eta, d\theta\} = \{d\eta, \theta\} = \{\eta, d\theta\} = 0\,. \tag{29.15}$$

Multiple Graßmann variables

- As an example, consider the "Gaussian" integral

$$\int d\eta d\theta e^{-\eta b\theta} = \int d\eta d\theta \left(1 - \eta b\theta\right) = b \int d\eta \eta \int d\theta \theta = b\,. \quad (29.16)$$

- We can generalize this result to multi-dimensional integrals, e.g.

$$\prod_i \int d\eta_i d\theta_i e^{-\eta_i B_{ij}\theta_j} = \mathrm{det} B_{ij} \quad (29.17)$$