Fermions

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Fall 2020

Review

- So far, we have dealt with QFT constructions for *scalar* fields
- These are spin-0 fields (bosons)
- However, most fundamental quantum fields in nature are *not* scalar fields; the exception is the Higgs field
- While there are scalar quantum fields in other contexts (e.g. cond-mat), let us now discuss how to set up spin-$\frac{1}{2}$ quantum fields (fermions)

Bosonic Ladder Operators

- In lecture 2, we considered the quantum mechanical Hamiltonian

$$\hat{\mathrm{H}} = -\frac{1}{2m}\partial_x^2 + \frac{m\omega^2\hat{x}^2}{2}\,. \tag{30.1}$$

- Using the rescaling of the operator

$$\hat{x} = \frac{\hat{q}}{\sqrt{m\omega}}\,, \tag{30.2}$$

  this can be written as $\hat{\mathrm{H}} = \frac{\omega}{2}\left(-\partial_q^2 + \hat{q}^2\right)$

- We can express this Hamiltonian as $\hat{\mathrm{H}} = \omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$ using the *ladder operators*

$$\hat{a} = \frac{\partial_q + \hat{q}}{\sqrt{2}}\,, \quad \hat{a}^\dagger = \frac{-\partial_q + \hat{q}}{\sqrt{2}}\,. \tag{30.3}$$

Bosonic Ladder Operators

- The ladder operators fulfill the commutation relations

$$[\hat{a}, \hat{a}^\dagger] = 1\,, \quad [\hat{a}, \hat{a}] = 0\,, \quad [\hat{a}^\dagger, \hat{a}^\dagger] = 0\,, \tag{30.4}$$

and can be used to build up energy eigenstates from

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle\,, \quad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle\,. \tag{30.5}$$

- We may use the *anti-commutator* $\{a, b\} = ab + ba$ to write the bosonic Hamiltonian as

$$\hat{\mathrm{H}} = \frac{\omega}{2}\left\{a^\dagger, a\right\}\,. \tag{30.6}$$

- This is useful for calculating the (bosonic) QM partition function

$$Z_B = \mathrm{Tr} e^{-\beta\hat{\mathrm{H}}} = \sum_{n=0}^{\infty}\langle n|e^{-\beta\hat{\mathrm{H}}}|n\rangle = \sum_{n=0}^{\infty} e^{-\beta\omega\left(n+\frac{1}{2}\right)} = \frac{1}{2\sinh\frac{\beta\omega}{2}}\,. \tag{30.7}$$

Fermionic Ladder Operators

- Let us now generalize the bosonic ladder operators to *fermionic* ladder operators
- Operationally, we do this by changing the *commutation relation* to an *anti-commutation relation*

$$\left\{\hat{a},\hat{a}^\dagger\right\}=1\,,\quad \left\{\hat{a},\hat{a}\right\}=0\,,\quad \left\{\hat{a}^\dagger,\hat{a}^\dagger\right\}=0\,. \tag{30.8}$$

- By analogy with (30.6), the *fermionic* Hamiltonian becomes

$$\hat{\mathrm{H}}=\frac{\omega}{2}[\hat{a}^\dagger,\hat{a}]=\omega\left(\hat{a}^\dagger\hat{a}-\frac{1}{2}\right)\,. \tag{30.9}$$

- The anti-commutation relation implies that the Hilbert space consists of only two energy eigenstates, $|0\rangle,|1\rangle$, because

$$a^\dagger|1\rangle=a^\dagger a^\dagger|0\rangle=\frac{1}{2}\left\{a^\dagger,a^\dagger\right\}|0\rangle=0\,. \tag{30.10}$$

Fermionic Ladder Operators

- Therefore, the *fermionic* QM partition function becomes

$$\begin{aligned} Z_F &= \sum_{n=0}^{n=1} \langle n | e^{-\beta \hat{\mathrm{H}}} | n \rangle = \langle 0 | e^{\frac{\beta\omega}{2}} | 0 \rangle + \langle 1 | e^{\frac{\beta\omega}{2} - \beta\omega a^\dagger a} | 1 \rangle \,, \\ &= e^{\frac{\beta\omega}{2}} + e^{-\frac{\beta\omega}{2}} \,, \\ &= 2\cosh\frac{\beta\omega}{2} \,. \end{aligned} \tag{30.11}$$

- This result should be compared to the bosonic QM partition function (30.7)

Bra- and Ket States

- We can define bra- and ket- states as eigenstates of $\hat{a}, \hat{a}^\dagger$ as

$$\begin{aligned} |c\rangle &\equiv e^{-c\hat{a}^\dagger}|0\rangle = \left(1 - c\hat{a}^\dagger\right)|0\rangle\,, \quad \hat{a}|c\rangle = c|0\rangle \\ \langle c| &\equiv \langle 0|e^{-\hat{a}c^*} = \langle 0|\left(1 - \hat{a}c^*\right)\,, \quad \langle c|\hat{a}^\dagger = \langle 0|c^*\,, \end{aligned}$$

  where c, c^* are Graßmann variables
- Such states possess the transition amplitude

$$\langle c'|c\rangle = \langle 0|\left(1 - \hat{a}c'^*\right)\left(1 - c\hat{a}^\dagger\right)|0\rangle = 1 + \langle 0|\hat{a}c'^* c\hat{a}^\dagger|0\rangle \qquad (30.12)$$

- Demanding that the fermionic ladder operators $\hat{a}, \hat{a}^\dagger$ also anti-commute with c, c^* we get
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$$\langle c'|c\rangle = 1 + \langle 0|c'^* c|0\rangle = 1 + c'^* c = e^{c'^* c}\,. \qquad (30.13)$$

Bra- and Ket States

- Using these definitions, we can write

$$\begin{aligned}\int dc^* dc e^{-c^* c} |c\rangle\langle c| &= \int dc^* dc e^{-c^* c} \left(1 - c\hat{a}^\dagger\right) |0\rangle\langle 0| \left(1 - \hat{a}c^*\right) ,\\ &= |0\rangle\langle 0| + \int dc^* dc c \hat{a}^\dagger |0\rangle\langle 0| \hat{a} c^* ,\\ &= |0\rangle\langle 0| + \int dc^* dc c |1\rangle\langle 1| c^* , &(30.14)\\ &= |0\rangle\langle 0| + |1\rangle\langle 1| = \mathbf{1} . &(30.15)\end{aligned}$$

- This generalizes the completeness relation $\mathbf{1} = \sum_n |n\rangle\langle n|$ from commuting systems to anti-commuting systems

Fermionic Trace

- Now consider the Graßmann integral

$$I_A = \int dc^* dc e^{-c^* c} \langle -c|\hat{\mathrm{A}}|c\rangle \,, \tag{30.16}$$

where $\hat{\mathrm{A}}$ is assumed to be a bosonic (commuting) operator

- We have

$$\begin{aligned} I_A &= \int dc^* dc(-c^* c)\langle 0|\hat{\mathrm{A}}|0\rangle - \int dc^* dc\langle 0|\hat{a} c^* \hat{\mathrm{A}} c \hat{a}^\dagger|0\rangle \,, \\ &= \langle 0|\hat{\mathrm{A}}|0\rangle - \langle 1|\int dc^* dc c^* \hat{\mathrm{A}} c|1\rangle \,, \qquad (30.17) \\ &= \langle 0|\hat{\mathrm{A}}|0\rangle + \langle 1|\hat{\mathrm{A}}|1\rangle \,, \\ &= \mathrm{Tr}\hat{\mathrm{A}} \,. \end{aligned}$$

Fermionic Partition Function

- We now have the technical tools to rewrite the fermionic partition function (30.11) as a path integral
- First use the trace relation (30.17) to write

$$Z_F = \mathrm{Tr} e^{-\beta \hat{\mathrm{H}}} = \int dc^* dc e^{-c^* c} \langle -c | e^{-\beta \hat{\mathrm{H}}} | c \rangle \,. \tag{30.18}$$

- Next, split the Boltzmann factor into a product of $N \gg 1$ pieces

$$e^{-\beta \hat{\mathrm{H}}} = e^{-\epsilon \hat{\mathrm{H}}} e^{-\epsilon \hat{\mathrm{H}}} e^{-\epsilon \hat{\mathrm{H}}} \ldots e^{-\epsilon \hat{\mathrm{H}}} \,, \tag{30.19}$$

where $\epsilon \equiv \frac{\beta}{N}$. Since $\hat{\mathrm{H}}$ commutes with itself, there are no commutators to consider, cf. Eq. (3.4)

Fermionic Path Integral

- Next, insert "unity" in the form of

$$\mathbf{1} = \int dc_i^* dc_i e^{-c_i^* c_i} |c_i\rangle\langle c_i| \,, \qquad (30.20)$$

in-between these different products.

- We get objects such as

$$\begin{aligned} e^{-c_i^* c_i}\langle c_i|e^{-\epsilon\hat{\mathrm{H}}}|c_{i-1}\rangle &= e^{-c_i^* c_i} e^{-\epsilon H(c_i^*,c_{i-1})}\langle c_i|c_{i-1}\rangle \,, \\ &= e^{-\epsilon\left[c_i^*\frac{(c_i-c_{i-1})}{\epsilon}+H(c_i^*,c_{i-1})\right]} \,, \qquad (30.21) \end{aligned}$$

- This takes care of most of the terms in (30.18), except for the left-most and right-most exponential

Fermionic Path Integral

- If we label $c = c_1$ in (30.18), this takes care of the right-most exponential
- For the left-most exponential, we thus have

$$
\begin{aligned}
&\int dc_1^* dc_1 e^{-c_1^* c_1} \langle -c_1 | e^{-\epsilon \hat{\mathrm{H}}} | \int dc_N^* dc_N | c_N \rangle \\
&= \int dc_1^* dc_1 \int dc_N^* dc_N e^{-c_1^* c_1 - \epsilon \mathrm{H}(-c_1^*, c_N)} \langle -c_1 | c_N \rangle \\
&= \int dc_1^* dc_1 \int dc_N^* dc_N e^{-c_1^* c_1 - c_1^* c_N - \epsilon \mathrm{H}(-c_1^*, c_N)} \\
&= \int dc_1^* dc_1 \int dc_N^* dc_N e^{-\epsilon \left[c_1^* \frac{c_1 + c_N}{\epsilon} + \mathrm{H}(-c_1^*, c_N) \right]}
\end{aligned}
\tag{30.22}
$$

which is of the same form as (30.21) if we identify

$$
c_{N+1} = -c_1 \,, \quad c_{N+1}^* = -c_1^* \,. \tag{30.23}
$$

Fermionic Path Integral

- We find that we can write the fermionic partition function as

$$Z_F = \int dc_N^* dc_N \int dc_{N-1}^* dc_{N-1} \ldots \int dc_1^* dc_1 e^{-S_E} \tag{30.24}$$

where we found the object S_E to be given by

$$S_E = \epsilon \sum_{i=1}^{N} \left[c_{i+1}^* \frac{c_{i+1} - c_i}{\epsilon} + H(c_{i+1}^*, c_i) \right] \tag{30.25}$$

- This is to be supplemented by the *anti-periodic* boundary conditions (30.23)

Continuum Fermionic QM Path Integral

- As in lecture 4, we can think of the N values $i = 1, 2, \ldots, N$ as points on a the thermal circle
- In the continuum limit $N \to \infty$ then c_i becomes a function of the imaginary time $\tau \in [0, \beta]$:

$$c_i \to c(\tau)\,, \quad c_i^* \to c^*(\tau)\,, \tag{30.26}$$

  with anti-periodic boundary conditions: $c(\beta) = -c(0)$, $c^*(\beta) = -c^*(0)$
- In the continuum limit then

$$Z_F = \int \mathcal{D}c^* \mathcal{D}c e^{-\int_0^\beta d\tau \left[c^* \frac{dc}{d\tau} + \mathrm{H}(c^*, c)\right]}\,. \tag{30.27}$$