The Dirac Equation

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Review

- In lecture 23 we calculated the classical equations of motions for a scalar field
$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} = 0\,. \tag{31.1}$$

- For a single, real, non-interacting scalar field $\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2$, the classical equations of motion become
$$\Box \phi - m^2 \phi = 0\,, \tag{31.2}$$
which is known as the Klein-Gordon equation
- In this lecture, we want to study a relativistic wave equation for *fermionic* fields

Schrödinger Equation

- To get started, recall that we have a wave equation for fermion (half-integer spin) fields: the Schrödinger equation
- For a free particle, the Schrödinger equation is given by

$$i\partial_t \psi = -\frac{1}{2m}\nabla^2\psi \tag{31.3}$$

- Unfortunately, since the time- and space derivatives appear asymmetrically, the Schrödinger equation is not relativistically invariant
- For a proper QFT we need relativistic invariance, hence we must generalize the Schrödinger equation

A dead end

- In the Schrödinger equation, the Hamiltonian is $\hat{\mathrm{H}} = \frac{p^2}{2m}$
- This looks very much like an expansion of the relativistic energy

$$E = \sqrt{p^2 + m^2} = m + \frac{p^2}{2m} + \ldots . \tag{31.4}$$

- It is tempting to write a wave equation such as

$$i\partial_t \psi = \sqrt{\hat{p}^2 + m^2}\psi \tag{31.5}$$

- Unfortunately, (31.5) is non-local and also not relativistically covariant; clearly, it cannot be correct

Dirac's trick

- However, iterating (31.5) leads to

$$-\partial_t^2\psi - \hat{p}^2\psi - m^2\psi = 0 = \Box\psi - m^2\psi\,, \tag{31.6}$$

  which is the Klein-Gordon equation
- So the "square" of (31.5) is a proper relativistic wave equation
- Dirac: let's try to take the "correct" square root of the Klein-Gordon equation
- Ansatz for Dirac equation:

$$i\partial_t\psi = \hat{H}\psi = \left(-i\vec{\alpha}\cdot\nabla + \beta m\right)\psi\,, \tag{31.7}$$

  with constant $\vec{\alpha}, \beta$

Dirac's trick

- Check if we can get Klein-Gordon equation from square:

$$
\begin{aligned}
-\partial_t^2\psi &= \left(-i\alpha_i\partial_i + \beta m\right)\left(-i\alpha_j\partial_j + \beta m\right)\psi\,,\\
&= \left[\alpha_i\alpha_j\partial_i\partial_j - i\left(\alpha_i\beta + \beta\alpha_i\right)\partial_i + \beta^2 m^2\right]\psi \qquad (31.8)
\end{aligned}
$$

- Clearly, for ordinary numbers $\vec{\alpha}, \beta$, this is *not* the Klein-Gordon equation because the linear derivative term does not vanish
- Dirac realized that it *can* become the Klein-Gordon equation if $\vec{\alpha}, \beta$ are *matrices* and in particular

$$
\{\alpha_i, \alpha_j\} = 2\delta_{ij}\mathbf{1}\,, \quad \{\alpha_i, \beta\} = 0\,, \quad \beta^2 = \mathbf{1}\,. \qquad (31.9)
$$

Dirac's trick

- In modern notation, we define the *Dirac* matrices $\gamma^\mu = (\beta, \beta\vec{\alpha})$
- The Dirac matrices fulfill the anti-commutation relations (Clifford algebra)

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}\mathbf{1} \tag{31.10}$$

- Under Hermitian conjugation, we have

$$\gamma^{0\dagger} = \gamma^0\,, \quad \gamma^{i\dagger} = -\gamma^i\,. \tag{31.11}$$

- Because of the Cliffor algebra (31.10), we may summarize Hermitian conjugation as

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0\,. \tag{31.12}$$

Dirac's trick

- The Dirac equation is given by

$$(i\partial_\mu \gamma^\mu - m)\,\psi = 0\,, \tag{31.13}$$

- The Dirac equation fulfills

$$(i\partial_\mu \gamma^\mu + m)\,(i\partial_\mu \gamma^\mu - m)\,\psi = \left(-\partial_\mu \partial_\nu \gamma^\mu \gamma^\nu - m^2\right)\psi \tag{31.14}$$

- Note that ψ, unlike ϕ, must have multiple components. We call ψ a **spinor field**

Dirac's matrices

- There are different choices for γ^μ fulfilling the Clifford algebra (31.10), called representations
- In 1+1 dimensions, we can e.g. choose from the Pauli matrices σ^i
- In 3+1 dimensions, we have for instance the Dirac representation

$$\gamma^0 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}\,, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\,. \tag{31.15}$$

- In 3+1 dimensions, since γ^μ are 4x4 matrices, the spinor field ψ also has four components, which is sometimes denoted by a greek index, e.g.

$$\psi = \psi_\alpha = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}\,. \tag{31.16}$$

Dirac Slash

- The combination $\partial_\mu \gamma^\mu$ appears often for spinor fields
- It is customary to introduce a new notation for this combination:

$$\not{\partial} \equiv \partial_\mu \gamma^\mu, \tag{31.17}$$

which is pronouned "slashed-d"

- This is called the *Dirac-slash*

Dirac Adjoint Spinor

- The Dirac spinor ψ is in general a four-component complex object
- We can obtain a real, scalar object by employing $\psi^\dagger$, the Hermitian adjoint of ψ:

$$\psi^\dagger \psi \tag{31.18}$$

- Because of the special role of the matrix $\beta = \gamma^0$, it is customary to define the *Dirac Adjoint*

$$\bar{\psi} \equiv \psi^\dagger \gamma^0 \,. \tag{31.19}$$