Fermions II

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Review

- In lecture 30 we calculated the fermionic path integral in QM

$$Z_F = \int \mathcal{D}c^* \mathcal{D}c e^{-\int_0^\beta d\tau \left[c^* \frac{dc}{d\tau} + \mathrm{H}(c^*, c)\right]} . \tag{32.1}$$

- Here $\hat{\mathrm{H}}(\hat{a}^\dagger, \hat{a})$ was the Hamiltonian built out of anti-commuting ladder operators, e.g.

$$\hat{\mathrm{H}} = \omega \left(\hat{a}^\dagger \hat{a} - \frac{1}{2}\right) , \quad \left\{\hat{a}, \hat{a}^\dagger\right\} = 1 . \tag{32.2}$$

- In lecture 31, we considered the Dirac field equations

$$\left(i\partial\!\!\!/ - m\right) \psi = 0 . \tag{32.3}$$

- In this lecture, we construct the Hamiltonian for the Dirac field, defining the path integral for half-integer spin fields

Lagrangian for Dirac Field

- In lecture 22, we constructed a Lagrangian density $\mathcal{L}$ for a complex scalar field ϕ based on symmetry (e.g. Lorentz transformations)
- In lecture 23, we found the equations of motion for ϕ, ϕ^*, which were treated as independent components
- For the Dirac field, we *have* the equations of motion: the Dirac equation (32.3)
- Treating ψ and it's Dirac adjoint $\bar{\psi}$ as independent components, the equations of motion

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{\psi}} = 0\,, \tag{32.4}$$

therefore must correspond to (32.3)

Lagrangian for Dirac Field

- The Lagrangian density that gives rise to the Dirac equation therefore is given by
$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi \tag{32.5}$$
- From the Lagrangian density, we can construct the Hamiltonian density $\mathcal{H}$ as
$$\mathcal{H} = \pi\partial_0\psi - \mathcal{L}\,. \tag{32.6}$$
- Here π is the conjugate momentum to the field ψ,
$$\pi = \frac{\partial\mathcal{L}}{\partial\partial_0\psi} = \bar{\psi}i\gamma^0 = i\psi^\dagger\,. \tag{32.7}$$
- We therefore find the Hamiltonian density given by
$$\mathcal{H} = \bar{\psi}\left(-i\gamma^i\partial_i + m\right)\psi \tag{32.8}$$

Hamiltonian for the Dirac Field

- From the Hamiltonian density, the Hamiltonian is given by

$$H = \int_x \mathcal{H} = \int_x \psi^\dagger \left(-i\gamma^0\gamma^i\partial_i + \gamma^0 m\right)\psi \tag{32.9}$$

- Here $\psi, \psi^\dagger$ are classical (but anti-commuting) fields
- If we were to *quantize* the theory, the operators corresponding to $\psi, \psi^\dagger$ would play the same role as the fermionic ladder operators $\hat{a}, \hat{a}^\dagger$
- Also, the form of the Hamiltonian (32.9) corresponds to (32.2), except for the constant term
- We therefore identify the classical (but anti-commuting) fields $\psi, \psi^\dagger$ with the classical (but anti-commuting) variables c, c^* in (32.1)

Path Integral for Fermions

- Using this identification, we can write down a path integral for the quantum field theory of a Dirac fermion
- We have

$$\mathcal{Z}_F = \int \mathcal{D}\psi^\dagger \mathcal{D}\psi e^{-S_E}\,, \tag{32.10}$$

where

$$S_E \equiv \int_0^\beta d\tau \int_x \psi^\dagger \left[\partial_\tau - i\gamma^0\gamma^i\partial_i + \gamma^0 m\right]\psi\,. \tag{32.11}$$

- As in lecture 5, we refer to S_E as the "Euclidean" action
- Note that the fermionic fields are *anti-periodic* in Euclidean time, e.g. $\psi(\beta, \mathbf{x}) = -\psi(0, \mathbf{x})$

Path Integral for Fermions

- We can define "Euclidean" Dirac matrices

$$\gamma_0^E = \gamma_0 \,, \quad \gamma_i^E = -i\gamma^i \,, \tag{32.12}$$

which obey

$$\left\{\gamma_a^E, \gamma_b^E\right\} = 2\delta_{ab}\mathbf{1} \,, \quad \gamma_a^{E\dagger} = \gamma_a^E \tag{32.13}$$

- We can then simplify the Euclidean Lagrangian density as

$$\mathcal{L}_E = \bar{\psi}\left[\gamma_a^E \partial_a + m\right]\psi \tag{32.14}$$

Path Integral for Fermions

- Finally, since $\psi^\dagger$ and $\bar{\psi}$ only differ by a (matrix)-constant, the Jacobian for changing $\mathcal{D}\psi^\dagger$ to $\mathcal{D}\bar{\psi}$ is just a constant
- Ignoring the constant Jacobian, we have

$$Z_F = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-\int_x \bar{\psi}\left[\gamma_a^E \partial_a + m\right]\psi}\,, \tag{32.15}$$

  as the path integral for a quantum field theory for a free Dirac fermion
- We will evaluate the partition function in the next lecture