Fermions III

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Review

- In lecture 32 we derived the QFT partition function for free Dirac fermions
$$Z_F = \int \mathcal{D}\bar\psi \mathcal{D}\psi e^{-\int_x \bar\psi\left[\gamma_a^E \partial_a + m\right]\psi}\,, \tag{33.1}$$

- Here γ_a^E are the Euclidean γ matrices and the fermions obey anti-periodic boundary conditions in the time-like direction,

$$\psi(\tau=\beta,\mathbf{x}) = -\psi(\tau=0,\mathbf{x})\,,\quad \bar\psi(\tau=\beta,\mathbf{x}) = -\bar\psi(\tau=0,\mathbf{x})\,. \tag{33.2}$$

- In this lecture, we will solve the path integral (33.1), mirroring the technique for bosonic fields in lecture 6

Fermionic Matsubara frequencies

- We start by writing the fermionic field as a Fourier series

$$\psi(\tau,\mathbf{x}) = \sum_{n=-\infty}^{\infty} e^{i\tilde{\omega}_n \tau}\psi(\omega_n,\mathbf{x})\,, \tag{33.3}$$

  cf. Eq. (6.5)

- Because the fermions are anti-periodic (33.2), the *fermionic* Matsubara frequencies are given by

$$\tilde{\omega}_n = \pi T(2n+1)\,. \tag{33.4}$$

- Note that unlike the bosonic Matsubara frequencies $\omega_n = 2\pi n T$, the fermionic Matsubara frequencies do not contain a zero mode

Fermionic Matsubara frequencies

- Similar to bosonic theories discussed in Eq. (9.5), we can also transform the spatial components of $\psi, \bar{\psi}$ to Fourier space:

$$\psi(\tau, \mathbf{x}) = \frac{T}{V} \sum_n \sum_{\mathbf{k}} e^{i\tilde{\omega}_n \tau + i\mathbf{k}\cdot\mathbf{x}} \psi(\omega_n, \mathbf{k}) \,, \tag{33.5}$$

  where V is the volume of space
- The boundary conditions for the fermions in the space-like directions will be un-important because of the infinite volume limit $V \to \infty$
- Taking the boundary conditions as periodic in space, anti-periodic in time, we can define a "fermionic" Euclidean momentum

$$\tilde{P}_a \equiv (\tilde{\omega}_n, \mathbf{p}) \tag{33.6}$$

Fermionic Action

- Using this notation, we have

$$\psi(x) = \frac{1}{\beta V}\sum_{\tilde{K}} e^{i\tilde{K}\cdot x}\psi(\tilde{K})\,, \quad \bar{\psi}(x) = \frac{1}{\beta V}\sum_{\tilde{K}} e^{-i\tilde{K}\cdot x}\bar{\psi}(\tilde{K})\,, \quad (33.7)$$

where V is the volume of space

- Plugging these expansions into the action in (33.1) we have

$$\begin{aligned} S_E &= \frac{1}{\beta^2 V^2}\int_x \sum_{\tilde{K},\tilde{P}} e^{ix\cdot(\tilde{P}-\tilde{K})}\bar{\psi}(\tilde{K})\left(i\gamma_a^E \tilde{P}_a + m\right)\psi(\tilde{P})\,, \\ &= \frac{1}{\beta V}\sum_{\tilde{P}} \bar{\psi}(\tilde{P})\left(i\gamma_a^E \tilde{P}_a + m\right)\psi(\tilde{P})\,, \\ &= \frac{1}{\beta V}\sum_{\tilde{P}} \bar{\psi}(\tilde{P})\left(i\tilde{\not{P}} + m\right)\psi(\tilde{P})\,. \end{aligned} \quad (33.8)$$

Fermionic Path Integral

- Up to an overall Jacobian from the Fourier-transformation (which we ignore), the fermionic partition function becomes

$$Z_F = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-\frac{1}{\beta V}\sum_{\tilde{P}} \bar{\psi}(\tilde{P})\left(i\tilde{\not{P}}+m\right)\psi(\tilde{P})} \qquad (33.9)$$

- This is a "Gaussian" integral for fermions, which are anti-commuting
- It can be evaluated using the result for multi-dimensional Graßmann integration given in (29.17):

$$\prod_i \int d\eta_i d\theta_i e^{-\eta_i B_{ij}\theta_j} = \det B_{ij}\,. \qquad (33.10)$$

Fermionic Path Integral

- We find

$$Z_F = \tilde{C} \prod_{\tilde{P}} \det \left[i\tilde{\not{P}} + m \right] , \tag{33.11}$$

where the determinant is over spinor space (spanned by the Dirac matrices γ_a^E) and $\tilde{C}$ is an overall constant.

- Since Z_F is real we can replace the determinant by

$$\begin{aligned} \det \left[i\tilde{\not{P}} + m \right] &= \det \sqrt{\left[-i\tilde{\not{P}} + m \right] \left[i\tilde{\not{P}} + m \right]} , \\ &= \det \sqrt{\left[\tilde{\not{P}}^2 + m^2 \right]} , \\ &= \det \sqrt{\left[\tilde{P}^2 + m^2 \right] \mathbf{1}_4} = \left[\tilde{P}^2 + m^2 \right]^2 . \end{aligned} \tag{33.12}$$

Fermionic Partition Function

- We have

$$Z_F = \tilde{C} \prod_{\tilde{P}} \left[\tilde{P}^2 + m^2\right]^2 = \tilde{C} \prod_{n=-\infty}^{\infty} \prod_{\vec{p}} \left[\tilde{\omega}_n^2 + E_p^2\right]^2 , \qquad (33.13)$$

  where $E_p = \sqrt{p^2 + m^2}$

- This bears some similarity to the partition function for a real scalar field, cf. (9.7), (9.10):

$$Z_B = C \prod_{P} \left[P^2 + m^2\right]^{-\frac{1}{2}} = C \prod_{n=-\infty}^{\infty} \prod_{\vec{p}} \left[\omega_n^2 + E_p^2\right]^{-\frac{1}{2}} . \qquad (33.14)$$

Fermionic Partition Function

- We evaluated the bosonic partition function in (6.20), (9.10)
- Ignoring the overall constant C, we had

$$Z_B = \prod_{\vec{p}} \frac{1}{2\sinh\left(\frac{\beta E_p}{2}\right)} = e^{-\sum_{\vec{p}}\left[\frac{\beta E_p}{2} + \ln\left(1-e^{-\beta E_p}\right)\right]}\,. \qquad (33.15)$$

- Similarly, we can use the result for the fermionic harmonic oscillator (30.11) to rewrite the fermionic partition function as

$$Z_F = \prod_{\vec{p}} \left[2\cosh\left(\frac{\beta E_p}{2}\right)\right]^4 = e^{4\sum_{\vec{p}}\left[\frac{\beta E_p}{2} + \ln\left(1+e^{-\beta E_p}\right)\right]}\,. \qquad (33.16)$$

Fermionic pressure

- Using the thermodynamic relation $p=\frac{\ln Z}{\beta V}$ we therefore find for the pressure of a free Dirac fermion

$$p_F=4\frac{1}{V}\sum_{\vec{p}}\left[\frac{E_p}{2}+T\ln\left(1+e^{-\beta E_p}\right)\right]\,, \tag{33.17}$$

- In the large volume limit $V\to\infty$, this becomes

$$p_F=4\int\frac{d^3p}{(2\pi)^3}\left[\frac{E_p}{2}+T\ln\left(1+e^{-\beta E_p}\right)\right]\,. \tag{33.18}$$

- For comparison, for the pressure of a free scalar field we had in (10.3):

$$p_B=-\int\frac{d^3p}{(2\pi)^3}\left[\frac{E_p}{2}+T\ln\left(1-e^{-\beta E_p}\right)\right]\,. \tag{33.19}$$

Fermionic pressure

- Focusing on the zero-temperature case $T = 0$ we find that $p_F = -4p_B$
- Recalling the concept of degrees of freedom, and the fact that the Dirac spinor has four components, we say that one fermionic d.o.f. is one-quarter of p_F
- Therefore, the pressure of one fermionic dof equals minus the pressure of one bosonic dof
- If we consider a theory of equal number of fermionic and bosonic dofs, these have *vanishing* pressure (cosmological constant)
- This is a key result in supersymmetry, which needs equal number of bosons and fermions