Gauge Fields

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Review

- So far, we have dealt with QFTs for scalar fields and fermions
- An important class of quantum fields found in nature are *gauge fields*
- In this lecture, we will start our discussion of gauge field QFTs

Complex Scalar Field

- In lecture 22, we discussed the action for a complex scalar field

$$S_E = \int_x \left[\partial_a \phi \partial_a \phi^* + V(\sqrt{\phi\phi^*})\right] . \tag{34.1}$$

- We found that – in addition to Lorentz invariance – the action (34.1) is also invariant under the transformation

$$\phi(x) \to e^{i\alpha}\phi(x) , \quad \left(\phi^*(x) \to e^{-i\alpha}\phi^*(x)\right) , \tag{34.2}$$

  with arbitrary (but constant) α
- Because α does not depend on x, we call this a **global** U(1) transformation

Global and Local Gauge Invariance

- The action (34.1) is invariant under a global U(1) transformation (34.2)
- We also call (34.2) a global *gauge transformation*
- If we make α dependent on x, the generalization of (34.2) is called a **local** gauge transformation:

$$\phi(x) \to e^{i\alpha(x)}\phi(x)\,, \quad \left(\phi^*(x) \to e^{-i\alpha(x)}\phi^*(x)\right)\,, \tag{34.3}$$

- We found that (34.1) is *not* invariant under local gauge transformations
- We can try to change the action such that it is locally invariant

Global and Local Gauge Invariance

- The "offending" term for the local gauge transformation is the first term in

$$\partial_a \phi \to e^{i\alpha(x)} \left[\phi \partial_a i\alpha(x) + \partial_a \phi\right] \tag{34.4}$$

- So how do we have to modify S_E in order to make the offending term disappear?
- We could add a new field A_a such that

$$S_E = \int_x \left[(\partial_a + iA_a)\,\phi\,(\partial_a - iA_a)\,\phi^* + V(\sqrt{\phi\phi^*})\right]\,. \tag{34.5}$$

- If now

$$A_a(x) \to A_a(x) - \partial_a \alpha(x) \tag{34.6}$$

along with (34.3), we find that (34.5) is invariant

Covariant Derivative

- We can define a new notation for the combination

$$\partial_a - iA_a \equiv D_a \,, \tag{34.7}$$

  and call D_a the *gauge-covariant derivative* (or covariant derivative for short)

- Using this notation, the action becomes

$$S_E = \int_x \left[D_a\phi \, (D_a\phi)^* + V(\sqrt{\phi\phi^*}) \right] \,, \tag{34.8}$$

  and is manifestly real, Lorentz-invariant, and locally gauge invariant.

- We call $A_a(x)$ the local U(1) gauge field

Covariant Derivative

- Since we have a new field $A_a(x)$, we may ask if there are terms that are real, Lorentz-invariant and gauge-invariant that are allowed in S_E
- The answer is affirmative
- Explicitly, consider the tensor

$$F_{ab} = \partial_a A_b - \partial_b A_a \,, \tag{34.9}$$

  which is a tensor under Lorentz-transformations
- It is straightforward to verify that F_{ab} is invariant under (34.3)
- We call F_{ab} the *field-strength tensor*

Covariant Derivative

- We find that we can add a piece to S_E such that

$$S_E = \int_x \left[D_a\phi \left(D_a\phi \right)^* + V(\sqrt{\phi\phi^*}) + \frac{1}{4} F_{ab} F_{ab} \right] , \qquad (34.10)$$

  where the factor $\frac{1}{4}$ is convention.
- Since all three contributions to S_E are real, and invariant under Lorentz and gauge transformations, so is (34.10)
- Another possible contribution is $F_{ab}\epsilon_{abcd}F_{cd}$ with ϵ_{abcd} the 4-dimensional Levi-Civita symbol;however, this contribution violates another symmetry (parity) which we'd like to have;for this reason, we do not allow this contribution here
- We call the (classical) theory that is defined by (34.10) *scalar electrodynamics*

Electromagnetism

- The theory is called scalar electrodynamics because the equations of motion for the gauge field $A_a(x)$ are nothing but Maxwell's equations
- The gauge field is coupled to the complex scalar field via the gauge-covariant coupling
- Therefore the complex scalar field corresponds to the *matter content* in this theory — it is the equivalent of the electron in real electromagnetism
- Since the electron is a fermion, scalar electromagnetism is not "real" electromagnetism; it's a "toy theory" which is a little simpler to analyze

Quantum Electrodynamics

- So far, we have dealt with the action (34.10) on a classical level
- We can aim for a QFT treatment of scalar electrodynamics by considering a path integral such as

$$Z = \int \mathcal{D}\phi^* \mathcal{D}\phi \mathcal{D}A e^{-S_E} \,. \tag{34.11}$$

- Unlike our results for scalar fields and fermions, we will encounter difficulties for the gauge fields A_a in this approach
- We will discuss this difficulties, and their resolution, in the following lectures