Quantum Electrodynamics

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Review

- In lecture 34, we found the action for scalar electrodynamics
- In this lecture, we focus on the gauge field part

$$S_E = \frac{1}{4}\int_x F_{ab}F_{ab}\,, \quad F_{ab} = \partial_a A_b - \partial_b A_a\,, \tag{35.1}$$

- We will aim for constructing a QFT partition function using the classical action S_E in this lecture

Naive Quantum Electrodynamics

- Proceeding as with scalar and fermionic field theories, we are tempted to write

$$Z = \int \mathcal{D}A e^{-S_E} \,. \tag{35.2}$$

- We can Fourier-transform the gauge fields as

$$A_a(x) = \frac{1}{\beta V} \sum_K e^{iK\cdot x} \tilde{A}_a(K) \,. \tag{35.3}$$

- The Euclidean action then becomes

$$S_E = \frac{1}{2\beta V} \sum_K \tilde{A}_a(K) \left[K^2 \delta_{ab} - K_a K_b \right] \tilde{A}_b(-K) \,. \tag{35.4}$$

Naive Quantum Electrodynamics

- The Jacobian from the Fourier transform is a constant and may again be neglected
- The resulting path integral is Gaussian and we find

$$Z = \prod_K \det{}^{-\frac{1}{2}} \left[K^2 \delta_{ab} - K_a K_b\right] \,. \tag{35.5}$$

- However, there is a problem with this result!
- The matrix $K^2\delta_{ab} - K_a K_b$ has a vanishing eigenvalue because

$$\left[K^2 \delta_{ab} - K_a K_b\right] K_a = 0 \,. \tag{35.6}$$

- Since there is a vanishing eigenvalue, the matrix is not invertible, and $\det\left[K^2\delta_{ab} - K_a K_b\right] = 0$

Naive Quantum Electrodynamics

- The naive version of quantum electrodynamics diverges
- It's easy to see why this happens
- Recall that S_E is invariant under gauge transformations:

$$A_a(x) \to A_a(x) - \partial_a \alpha(x)\,, \tag{35.7}$$

  for any function $\alpha(x)$
- In our path integral, we integrate over all $A_a(x)$, hence also over all those that equivalent under the gauge transformation (35.7)
- Since S_E is constant for all gauge-equivalent A's, we have

$$\int_{-\infty}^{\infty} d\alpha e^{\mathrm{const}}\,, \tag{35.8}$$

  as part of Z, which diverges

Compact Quantum Electrodynamics

- There is a (non-perturbative) way to make sense of this theory:
- We could regulate the divergence properly, so that it does not affect expectation values, e.g. by compactifying the range of the gauge parameter:

$$\int_{-\infty}^{\infty} d\alpha \to \int_{-\Lambda}^{\Lambda} d\alpha\,. \tag{35.9}$$

- The resulting theory is known as *Compact U(1)* gauge theory, and can be studied e.g. using lattice gauge theory
- It has interesting properties (self-interacting photons) that do not seem (?) to correspond to what happens in nature
- For this reason, we will study a *different* way to repair the theory

Fadeev-Popov Trick

- The problem of the divergent Z is a direct result of invariance under gauge-transformations
- There is an easy, albeit not very elegant, way out: break gauge invariance
- Specifically, we can choose a gauge condition $G[A]=0$ that fixes the gauge. Examples for this are $G[A]=A_0$ (temporal axial gauge) or $\partial_i A_i = 0$ (Coulomb gauge) or $\partial_a A_a = 0$ (Landau gauge)
- Start by considering a path integral for Z *only* over *inequivalent* gauge fields $\bar{A}$:

$$Z = \int \mathcal{D}\bar{A} e^{-S_E[\bar{A}]} \tag{35.10}$$

- We can "stick-in" unity in this expression by writing

$$Z = \int \mathcal{D}\bar{A}\mathcal{D}G\delta(G[A]) e^{-S_E[\bar{A}]} \tag{35.11}$$

Fadeev-Popov Trick

- Changing variables for G to the gauge parameter $\alpha(x)$ we have

$$Z = \int \mathcal{D}\bar{A}\mathcal{D}\alpha\delta(G[A])\det\left(\frac{\partial G[A]}{\partial\alpha}\right)e^{-S_E[\bar{A}]} \tag{35.12}$$

- But the path integral over gauge-inequivalent fields $\bar{A}_a$ and all gauge parameters α is the same as the path integral over all gauge fields:

$$Z = \int \mathcal{D}A\delta(G[A])\det\left(\frac{\partial G[A]}{\partial\alpha}\right)e^{-S_E[\bar{A}]} \tag{35.13}$$

- With the δ-function restricting S_E to gauge-inequivalent values, we may write

$$Z = \int \mathcal{D}A\delta(G[A])\det\left(\frac{\partial G[A]}{\partial\alpha}\right)e^{-S_E[A]} \tag{35.14}$$

Fadeev-Popov Trick

- While $G[A] = 0$ does the trick, we could just as well use $G[A] = f$, with f an arbitrary (A-independent) function
- So we can replace $\delta(G[A])$ by $\delta(G[A] - f)$
- Since any function f does the trick, we can average over *all* f
- Performing a path-integral average with a Gaussian weight for f then leads to

$$Z_{gf} = \int \mathcal{D}A\mathcal{D}f\,\delta(G[A] - f)\mathrm{det}\left(\frac{\partial G[A]}{\partial \alpha}\right) e^{-\frac{1}{2\xi}\int_x f^2(x)} e^{-S_E}\,, \quad (35.15)$$

  where ξ is an arbitrary parameter
- Using the δ-function, we can perform the integral over f and find

$$Z_{gf} = \int \mathcal{D}A\mathrm{det}\left(\frac{\partial G[A]}{\partial \alpha}\right) e^{-\frac{1}{2\xi}\int_x G^2[A]} e^{-S_E}\,. \quad (35.16)$$

Fadeev-Popov Trick

- We now have a path integral over an exponential that looks very similar to the ones for scalars/fermions we had before
- The only sore is the determinant
- We can *formally* write the determinant as a path integral over an exponential by exploiting integration over Grassmann fields $\bar{c}, c$:

$$Z = \int \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c e^{-\frac{1}{2\xi}\int_x G^2[A] - S_E - \int_x \bar{c}\frac{\partial G[A]}{\partial\alpha}c} . \tag{35.17}$$

- The fields $\bar{c}, c$ are called **Faddeev-Popov ghosts** because they are not "real" fields, but merely introduced as a mathematical trick
- *Unlike fermion fields*, the ghosts fulfill *periodic* boundary conditions, just like scalar fields!

QED

- We will solve the gauge-fixed partition function in the next lecture
- Including matter fields such as the complex scalar field ϕ, ϕ^* or an electron spinor ψ does not change the procedure
- As a consequence, we find that quantum electrodynamics can be defined by

$$Z = \int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\bar{c}\mathcal{D}c e^{-S_{\rm matter}-S_{\rm gauge}-S_{\rm gf}-S_{\rm ghosts}}\,, \qquad (35.18)$$

where $S_{\rm matter}$ is the *matter part*, $S_{\rm gauge} = \frac{1}{4}\int_x F_{ab}^2$ is the *gauge field part*, $S_{\rm gf} = \frac{1}{2\xi}\int_x G^2[A]$ is the *gauge-fixing* part and $S_{\rm ghosts} = \int_x \bar{c}\frac{\partial G}{\partial \alpha}c$ is the *ghost* part of the theory