Solving the U(1) Path Integral

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Review

- In lecture 35, we found the gauge-fixed partition function for a pure U(1) gauge field:

$$Z = \int \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c e^{-S_{\rm gauge}-S_{\rm gf}-S_{\rm ghost}}\,. \tag{36.1}$$

- Here

$$S_{\rm gauge} = \frac{1}{4}\int_x F_{ab}^2\,,\quad S_{\rm gf} = \frac{1}{2\xi}\int_x G^2[A]\,,\quad S_{\rm ghost} = \int_x \bar{c}\frac{\partial G[A]}{\partial \alpha}c\,, \tag{36.2}$$

  where $F_{ab} = \partial_a A_b - \partial_b A_a$, $G[A]$ an arbitrary gauge-fixing condition, and ξ an arbitrary gauge-fixing parameter

- Let us solve the partition function in this lecture

Gauge-Fixing

- To get started, we have to choose a gauge condition $G[A]$
- All choices for $G[A]$ must lead to the same result, but some choices lead to easier calculations of Z than others
- Let's start with the canonical choice of Landau gauge:

$$G[A] = \partial_a A_a \,. \tag{36.3}$$

- Since $A_a \to A_a - \partial_a \alpha$ under gauge transformations, this immediately gives

$$S_{\rm ghost} = \int_x \bar{c} \frac{\partial G[A]}{\partial \alpha} c = \int_x \bar{c} \partial_a \frac{A_a}{\partial \alpha} c = \int_x \partial_a \bar{c} \partial_a c \tag{36.4}$$

Gauge-Fixing

- The ghost action does not depend on A_a and $S_{\rm gauge}+S_{\rm gf}$ do not depend on the ghosts
- For the U(1) gauge field, the partition function separates:

$$Z = Z_{\rm A}\times Z_{\rm ghost}\,, \tag{36.5}$$

- Here

$$Z_{\rm A}=\int \mathcal{D}A e^{-S_{\rm gauge}-S_{\rm gf}}\,,\quad Z_{\rm ghost}=\int \mathcal{D}\bar{c}\mathcal{D}c e^{-S_{\rm ghost}}\,. \tag{36.6}$$

Periodicity

- As was the case for the scalar fields and fermions, we start by Fourier-transforming the fields A:

$$A_a(x) = \frac{1}{\beta V} \sum_K e^{iK\cdot x} \tilde{A}_a(x) \,. \tag{36.7}$$

- Recall that we introduced $A_a(x)$ as necessary to make scalar QED invariant under the local gauge transformations

$$\phi(x) \to e^{i\alpha(x)}\phi(x) \,, \quad A_a(x) \to A_a(x) - \partial_a \alpha(x) \,. \tag{36.8}$$

- Since the scalar $\phi(x)$ was periodic in imaginary time $\phi(\tau = \beta, \mathbf{x}) = \phi(\tau = 0, \mathbf{x})$, this implies that the gauge-transformation α also must be periodic
- As a consequence, the gauge fields $A_a(x)$ are also periodic in the time-like direction, and $K_0 = \omega_n = 2\pi n T$

Path Integral

- We obtain for the Fourier-transformed partition function

$$Z_{\rm A} = \int \mathcal{D}\tilde{A} e^{-\frac{1}{2\beta V}\sum_K \tilde{A}_a(K)\left[K^2\delta_{ab} - K_a K_b + \frac{1}{\xi}K_a K_b\right]\tilde{A}_b(-K)} \tag{36.9}$$

- The path-integral is Gaussian, so we obtain

$$Z_{\rm A} = \prod_K \det{}^{-\frac{1}{2}}\left[K^2\delta_{ab} - K_a K_b + \frac{1}{\xi}K_a K_b\right] \tag{36.10}$$

- We need to calculate the determinant of the matrix $M_{ab} \equiv \left[K^2\delta_{ab} - K_a K_b + \frac{1}{\xi}K_a K_b\right]$

Matrix Determinant

- We can decompose the Matrix using the two projectors

$$P^T_{ab} = \delta_{ab} - \frac{K_a K_b}{K^2}\,, \quad P^L_{ab} = \frac{K_a K_b}{K^2}\,, \tag{36.11}$$

- These projectors obey

$$P^T_{ab} P^L_{bc} = 0\,, \quad P^T_{ab} P^T_{bc} = P^T_{ac}\,, \quad P^L_{ab} P^L_{bc} = P^L_{ac}\,. \tag{36.12}$$

- In terms of these projectors we have

$$M_{ab} = K^2 P^T_{ab} + \frac{K^2}{\xi} P^L_{ab} \tag{36.13}$$

- This implies that the eigenvalues of M_{ab} are K^2 and K^2/ξ, respectively

Matrix Determinant

- Noting furthermore that

$$\mathrm{Tr} P^T_{ab} = \delta_{aa} - 1 = 3\,, \quad \mathrm{Tr} P^L_{ab} = 1\,, \tag{36.14}$$

  we find that the K^2 eigenvalue has multiplicity 3, and K^2/ξ has multiplicity one
- This gives

$$\det M_{ab} = \left(K^2\right)^3 \left(\frac{K^2}{\xi}\right)^1\,, \tag{36.15}$$

- As a consequence, we have

$$Z_{\rm A} = e^{-\frac{1}{2}\sum_K \ln\left[K^2\right]^4 + \frac{1}{2}\sum_K \ln[\xi]}\,. \tag{36.16}$$

Ghost Part

- For the ghost part of the action, transformation to Fourier space leads to

$$
\begin{aligned}
Z_{\rm ghost} &= \int \mathcal{D}\bar{c}\mathcal{D}c e^{-\frac{1}{\beta V}\sum_K \bar{c}(K)K^2 c(K)}\,,\\
&= \prod_K K^2\,,\\
&= e^{\sum_K \ln K^2}\,. \qquad (36.17)
\end{aligned}
$$

U(1) Partition Function

- Combining all parts, we have

$$Z = Z_{\mathrm{A}} \times Z_{\mathrm{ghost}} = e^{-\frac{1}{2}\sum_K \ln\left[K^2\right]^4 + \frac{1}{2}\sum_K \ln[\xi] + \sum_K \ln K^2}\,. \tag{36.18}$$

- In the large volume limit, the sums become

$$\sum_K = \frac{1}{V}\sum_{\omega_n}\int \frac{d^3k}{(2\pi)^3}\,. \tag{36.19}$$

- In dimensional regularization, the integral over a constant vanishes because there is no logarithmic divergence
- Hence

$$\frac{1}{2}\sum_K \ln\left[\xi\right] \to 0\,, \tag{36.20}$$

in dim-reg.

U(1) Partition Function

- Of the remaining parts, we have

$$Z = e^{-\frac{1}{2}\sum_K \ln\left[K^2\right]^4 + \sum_K \ln K^2}\,. \tag{36.21}$$

- We see that the contribution from the ghosts cancels half of the contribution from the gauge fields
- We find

$$Z = e^{-\frac{1}{2}\sum_K \ln\left[K^2\right]^2} = e^{-\frac{1}{2}\times 2\sum_K \ln\left[\omega_n^2+\mathbf{k}^2\right]} \tag{36.22}$$

- Comparing to Eq. (33.14), this is exactly equal to the partition function of two free, real, massless scalar fields

U(1) Partition Function

- We find for the pressure of a U(1) gauge field

$$p(T) = 2p_{\rm free}(m=0,T) = 2\times\frac{\pi^2 T^4}{90}\,. \tag{36.23}$$

- This is the pressure for perfect *blackbody radiation*
- We note that the original gauge field A_a had four degrees of freedom, which matches our result for Z_A
- However, the ghosts contributed *minus* two degrees of freedom, which left us with two physical degrees of freedom for the U(1) gauge field