Non-Abelian Gauge Theories

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Review

- In lecture 34, we considered the action for a complex scalar field

$$S_E = \int_x \left[(\partial_a + iA_a)\,\phi\,(\partial_a + iA_a)^*\,\phi^* + m^2 \phi^* \phi \right] \tag{37.1}$$

- We found that the action is invariant under *local* U(1) gauge transformations if

$$\phi(x) \to e^{i\alpha(x)}\phi(x)\,, \quad A_a(x) \to A_a(x) - \partial_a \alpha(x)\,. \tag{37.2}$$

- We know from lecture 26 that the U(1) transformation is isomorphic to SO(2) if we write $\phi = \frac{1}{\sqrt{2}}\left(\phi_1 + i\phi_2\right)$
- Let us now investigate how to generalize this from SO(2) to SO(N)

SO(3)

- To this end, consider an 3-component vector model

$$S_E = \int_x \left[\frac{1}{2} \partial_a \vec{\phi} \cdot \partial_a \vec{\phi} + \frac{m^2}{2} \vec{\phi} \cdot \vec{\phi} \right] . \tag{37.3}$$

- The action is invariant under a *global* SO(3) transformation

$$\phi_i(x) \to R_{ij}(\alpha_1, \alpha_2) \phi_j(x) \,, \tag{37.4}$$

  where R_{ij} is a rotation matrix in 3 dimensions w.r.t. 2 *constant* angles α_1, α_2
- While we can carry through the analysis using SO(3), it's slightly more convenient to switch to use SU(2) instead

Isomorphism SO(3)/SU(2)

- The action for a complex scalar field and a 2-component vector field is the same
- We can use a generalized version for the 3-component vector field
- To this end, consider the object

$$\Phi(x) = \frac{1}{2}\sum_{i=1}^{3}\phi_i(x)\sigma_i\,, \tag{37.5}$$

  where $\sigma_{1,2,3}$ are the Pauli-matrices
- The Pauli matrices fulfill $\mathrm{Tr}[\sigma_i\sigma_j] = 2\delta_{ij}$ and $\sigma_i^\dagger = \sigma_i$ so that

$$\mathrm{Tr}\left[\partial_a\Phi(x)\partial_a\Phi^\dagger(x)\right] = \frac{1}{4}\partial_a\phi_i(x)\partial_a\phi_j(x)\mathrm{Tr}\left[\sigma_i\sigma_j\right] = \frac{1}{2}\partial_a\phi_i(x)\partial_a\phi_i(x) \tag{37.6}$$

Isomorphism SO(3)/SU(2)

- The action (37.3) therefore can be written as

$$S_E = \mathrm{Tr} \int_x \left[\partial_a \Phi \partial_a \Phi^\dagger + m^2 \Phi \Phi^\dagger \right] . \tag{37.7}$$

- This action is invariant under the *global* SU(2) symmetry

$$\Phi(x) \to e^{i\alpha_i \sigma_i} \Phi(x) , \quad \Phi^\dagger(x) \to \Phi^\dagger(x) e^{-i\alpha_i \sigma_i} \tag{37.8}$$

  because

$$\mathrm{Tr} \left[e^{i\alpha_i\sigma_i} \Phi(x) \Phi^\dagger(x) e^{-i\alpha_i\sigma_i} \right] = \mathrm{Tr} \left[e^{-i\alpha_i\sigma_i} e^{i\alpha_i\sigma_i} \Phi(x) \Phi^\dagger(x) \right] .$$

Isomorphism SO(3)/SU(2)

- Let's now consider how to make an action that is invariant under *local* SU(2) gauge transformations:

$$\Phi(x) \to e^{i\alpha_i(x)\sigma_i}\Phi(x)\,, \tag{37.9}$$

- Taking our knowledge from the U(1) case (37.1), we are tempted to write

$$S_E = \mathrm{Tr}\int_x \left[D_a\Phi\left(D_a\Phi\right)^\dagger + m^2\Phi\Phi^\dagger\right]\,,\quad D_a \equiv \partial_a + iA_a(x)\,. \tag{37.10}$$

- Here the gauge field needs to transform as

$$A_a(x) \to e^{i\alpha_i(x)\sigma_i}A_a(x)e^{-i\alpha_i(x)\sigma_i} + i\left(\partial_a e^{i\alpha_i(x)\sigma_i}\right)e^{-i\alpha_i(x)\sigma_i} \tag{37.11}$$

Isomorphism SO(3)/SU(2)

- Let's introduce the new short-hand notation

$$U(x) \equiv e^{i\alpha_i(x)\sigma_i}\,, \tag{37.12}$$

- With this, the SU(2) gauge field transformation is

$$A_a(x) \to U(x)A_a(x)U^\dagger(x) + i\left(\partial_a U(x)\right)U^\dagger(x)\,. \tag{37.13}$$

and

$$D_a\Phi(x) \to U(x)D_a\Phi(x) \tag{37.14}$$

- What is the equivalent of F_{ab} for the SU(2) gauge field?

Non-Abelian Field-Strength Tensor

- For the U(1) gauge field, we can write

$$F_{ab} = -i\,[D_a, D_b] = \partial_a A_b - \partial_b A_a\,, \tag{37.15}$$

where $[.,.]$ denotes the commutator
- Let's try this for SU(2):

$$F_{ab} = -i\,[D_a, D_b] = \partial_a A_b - \partial_b A_a + i\,[A_a, A_b]\;. \tag{37.16}$$

- Does it transform correctly? Yes (homework problem):

$$F_{ab} \to U F_{ab} U^\dagger\,. \tag{37.17}$$

Non-Abelian Action

- A term such as $\mathrm{Tr}\left[F_{ab}F_{ab}\right]$ is invariant
- Include in the action:

$$S_E = \mathrm{Tr}\int_x \left[D_a\Phi\left(D_a\Phi\right)^\dagger + m^2\Phi\Phi^\dagger + \frac{1}{2g^2}F_{ab}F_{ab}\right], \qquad (37.18)$$

  where the constant factor $\frac{1}{2g^2}$ is convention
- We will call g the coupling constant for the non-abelian field theory

Non-Abelian Action

- As with the scalar field (37.5), we can decompose the field-strength tensor in components

$$F_{ab}(x) = \frac{1}{2}\sum_{i=1}^{3} \sigma_i F^i_{ab}\,. \tag{37.19}$$

- In this representation then

$$\mathrm{Tr}\left[\frac{1}{2g^2}F_{ab}F_{ab}\right] = \frac{1}{4g^2}F^i_{ab}F^i_{ab}\,, \tag{37.20}$$

where there is summation over both the Euclidean Lorentz indices $a, b = 1, 2, 3, 4$ and the "color" index $i = 1, 2, 3$

Non-Abelian Action

- Ignoring the matter part, we are left with the classical action for an SU(2) gauge field:

$$S_E = \frac{1}{2g^2}\mathrm{Tr}\int_x F_{ab}F_{ab}\,, \quad F_{ab} = \partial_a A_b - \partial_b A_a + i\left[A_a, A_b\right]\,. \quad (37.21)$$

- We derived this result for SU(2) gauge fields (37.19)
- However, we can *generalize* (37.21) to gauge groups SU(N) with arbitrary N
- For SU(N), (37.21) remains invariant for

$$A_a(x) \to U(x)A_a(x)U^\dagger(x) + i\left(\partial_a U(x)\right)U^\dagger(x)\,, \quad (37.22)$$

and $U(x) = e^{i\alpha_i(x)\lambda_i}$, where λ_i, $i \in [1, N^2-1]$ is an element of SU(N)

SU(N) Structure Constants

- We can represent the non-abelian field-strength tensor in components by using the *generators* t_i,

$$F_{ab} = \sum_{i=1}^{N^2-1} F^i_{ab} t_i \tag{37.23}$$

normalized such that $\mathrm{Tr}(t_i t_j) = \frac{1}{2}\delta_{ij}$.

- For SU(N), the generators are $\mathrm{N} \times \mathrm{N}$ matrices that obey the commutation relation

$$[t_i, t_j] = i f_{ijk} t_k \, , \tag{37.24}$$

where f_{ijk} are the *structure constants* of the SU(N) Lie algebra

- For N=2, $t_i = \frac{1}{2}\sigma_i$ and $f_{ijk} = \epsilon_{ijk}$.

Representation

- While F_{ab} and F^i_{ab} encode the same information, they correspond to different *representations*
- In the fundamental representation, one can think of as F_{ab} as a $\mathrm{N} \times \mathrm{N}$ matrix
- By contrast, F^i_{ab} is the *adjoint* representation, which corresponds to the $N^2 - 1$ independent entries of the fundamental representation
- We can use the properties of the generators to derive the field-strength tensor in the adjoint representation as

$$F^i_{ab} = 2\mathrm{Tr}\left[F_{ab} t^i\right] = \partial_a A^i_b - \partial_b A^i_a - f_{ijk} A^j_a A^k_b \,. \qquad (37.25)$$

Yang-Mills

- For general N, the SU(N) gauge theory is referred to as (classical) Yang-Mills theory
- For the special case N=3, the gauge theory becomes relevant for the theory of strong interactions, QCD
- For the special case N=2, the gauge theory becomes relevant for the electroweak theory (Salam-Glashow-Weinberg model)