Solving Non-Abelian Gauge Theories

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Review

- In lecture 37, we considered the action for the non-abelian gauge group SU(N):

$$S_E = \frac{1}{2g^2}\mathrm{Tr}\int_x [F_{ab}F_{ab}] = \frac{1}{4g^2}\int_x F^i_{ab}F^i_{ab} \tag{38.1}$$

where

$$F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a - f_{ijk}A^j_a A^k_b\,. \tag{38.2}$$

- The sum in the integrand is over both Lorentz indices a, b and color indices $i \in [1, N^2 - 1]$
- In this lecture, we will discuss the *quantum field theory* for the action (38.1)

Partition function for SU(N) Gauge Theory

- The partition function for SU(N) gauge theory is

$$Z = \int \mathcal{D}A e^{-\frac{1}{4g^2}\int_x F^i_{ab}F^i_{ab}}\,, \tag{38.3}$$

- The partition function is a function of the gauge coupling g
- We can always rescale the fields A as

$$A_a(x) \to g A_a(x)\,, \tag{38.4}$$

such that

$$Z = \int \mathcal{D}A e^{-\frac{1}{4}\int_x F^i_{ab}F^i_{ab}}\,, \quad F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a - g f_{ijk} A^j_a A^k_b\,. \tag{38.5}$$

Partition function for SU(N) Gauge Theory

- In this form it is clear that for $g \to 0$, the partition function for SU(N) gauge theory factorizes:

$$Z[g=0] = \prod_{i=1}^{N^2-1} \int \mathcal{D}A e^{-\frac{1}{4}\int_x F^2} = Z_{U(1)}^{N^2-1}\,, \qquad (38.6)$$

  where here $F^2 = \partial_a A_b - \partial_b A_a$
- Therefore, the *free* Yang-Mills theory corresponds to N^2-1 copies of electromagnetism
- As a consequence, free Yang-Mills theory (and hence weak-coupling perturbation theory) suffers from the same gauge-orbit problems as U(1)
- *Unlike* electromagnetism, the Yang-Mills partition function (38.5) is well defined for $g \neq 0$

Perturbative Yang-Mills Theory

- While Yang-Mills theory is non-perturbatively well-defined, it is hard to evaluate $Z[g]$ in practice
- We can nevertheless apply the Faddeev-Popov program from electromagnetism in order to do perturbation theory near $g = 0$
- We again start by considering the path-integral restricted over gauge-inequivalent field configurations $\bar{A}$:

$$Z[g] = \int \mathcal{D}\bar{A} e^{-S_E} \,. \tag{38.7}$$

- Now put in unity by restricting to some gauge condition $G^i[A] = f^i$,

$$Z[g] = \int \mathcal{D}\bar{A} \mathcal{D}G \delta(G^i[A] - f^i) e^{-S_E} \,. \tag{38.8}$$

Gauge Condition

- For the Yang-Mills field, examples for the gauge condition are

$$G^i[A] = \partial_a A^i_a\,, \quad G^i[A] = A^i_0\,, \quad \text{etc.} \tag{38.9}$$

- Changing variables from G to the gauge parameter α, and recognizing $\mathcal{D}\bar{A}\mathcal{D}\alpha = \mathcal{D}A$ we have

$$Z[g] = \int \mathcal{D}A\delta(G^i[A] - f^i)\mathrm{det}\left(\frac{\partial G^i[A]}{\partial \alpha^j}\right) e^{-S_E}\,. \tag{38.10}$$

- Writing the determinant as an exponential with the help of the Grassmann fields $\bar{c}^i, c^i$ we have

$$Z[g] = \int \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c\delta(G^i[A] - f^i)e^{-S_E - \int_x \bar{c}^i \frac{\partial G^i[A]}{\partial \alpha^j} c^j}\,. \tag{38.11}$$

Gauge-Fixed Partition Function

- Integrating over f^i with Gaussian weight leads to

$$Z[g] = \int \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c e^{-S_E - S_{\rm ghost} - S_{\rm gf}}\,. \tag{38.12}$$

where

$$S_E = \frac{1}{4g^2}\int_x F^i_{ab}F^i_{ab}\,,$$

$$S_{\rm ghost} = \int_x \bar{c}^i \frac{\partial G^i[A]}{\partial \alpha^j} c^j\,, \tag{38.13}$$

$$S_{\rm gf} = \frac{1}{2\xi}\int_x G^i[A]G^i[A]\,. \tag{38.14}$$

Example in Landau Gauge

- As an example, consider Landau gauge

$$G^i[A] = \partial_a A_a^i \,. \tag{38.15}$$

- In the fundamental representation, the gauge field transformed as

$$A_a(x) \to U(x) A_a(x) U^\dagger(x) + i \left(\partial_a U(x)\right) U^\dagger(x) \,. \tag{38.16}$$

under gauge transformations $U(x) = e^{i\alpha^i(x) t^i}$

- For small α, this gives in the adjoint representation

$$A_a^i(x) \to A_a^i(x) - \partial_a \alpha^i(x) - f^{ijk} \alpha^j(x) A_a^k(x) \,. \tag{38.17}$$

Example in Landau Gauge

- Unlike the case for U(1), the gauge-transformation for A_a depends on the field itself
- As a consequence, in the Landau gauge the ghost part of the action becomes

$$S_{\rm ghost} = \int_x \bar{c}^i \frac{\partial G^i[A]}{\partial \alpha^j} c^j = \int_x \left[\partial_a \bar{c}^i \partial_a c^j + \partial_a \bar{c}^i f^{ijk} A_a^k c^j\right] \qquad (38.18)$$

- Rescaling again $A_a(x) \to g A_a(x)$ then gives the gauge-fixed partition function in the Landau gauge:

$$Z = \int \mathcal{D}A \mathcal{D}\bar{c} \mathcal{D}c e^{-\int_x \left[\frac{1}{4} F_{ab}^i F_{ab}^i + \frac{1}{2\xi} \partial_a A_a^i \partial_a A_a^i + \partial_a \bar{c}^i \partial_a c^j + g \partial_a \bar{c}^i f^{ijk} A_a^k c^j\right]} . \qquad (38.19)$$

Example in Landau Gauge

- Unlike in electromagnetism, there is a self-interaction for the gauge fields because $F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a - f^{ijk} A^j_a A^k_b$ is nonlinear in the field
- Because there is now a term coupling the ghosts $\bar{c}, c$ and the gauge field A_a whenever $g \neq 0$, the ghosts no longer decouple
- When calculating observables in Yang-Mills theory, we generally have to deal with gauge-field vertices as well as ghost-gauge field vertices
- Even one-loop calculations are generally non-trivial in Yang-Mills theory!