Encore: Solving the U(1) Partition Function in TAG

paul.romatschke@colorado.edu

Fall 2020

Review

- In lecture 35, we defined the partition function for a pure U(1) gauge field:

$$Z=\int \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c e^{-S_{\rm gauge}-S_{\rm gf}-S_{\rm ghost}}\,. \tag{39.1}$$

- Here

$$S_{\rm gauge}=\frac{1}{4}\int_x F_{ab}^2\,,\quad S_{\rm gf}=\frac{1}{2\xi}\int_x G^2[A]\,,\quad S_{\rm ghost}=\int_x \bar{c}\frac{\partial G[A]}{\partial\alpha}c\,, \tag{39.2}$$

  where $F_{ab}=\partial_a A_b-\partial_b A_a$, $G[A]$ an arbitrary gauge-fixing condition, and ξ an arbitrary gauge-fixing parameter
- In lecture 36, we solved (39.1) for the class of covariant gauges, $G[A]=\partial_a A_a$
- In this lecture, we repeat the calculation for the temporal axial gauge, $G[A]=-A_0$ (the sign is convention)

Gauge condition and Ghost action

- In the TAG, because $A_a \to A_a - \partial_a \alpha$ under gauge transformations, we have

$$S_{\rm ghost} = \int_x \bar{c} \partial_0 c \, . \qquad (39.3)$$

- Since the ghost action does not depend on the fields A, the partition function again separates as in lecture 36:

$$Z = Z_A \times Z_{\rm ghost} \, , \quad Z_{\rm ghost} = \int \mathcal{D}\bar{c} \mathcal{D} c e^{-S_{\rm ghost}} \, . \qquad (39.4)$$

- The gauge-field part is

$$Z_A = \int \mathcal{D} A e^{-\frac{1}{4}\int_x F_{ab}F_{ab} - \frac{1}{2\xi}\int_x A_0^2} \, . \qquad (39.5)$$

Gauge Parameter

- We can perform the path-integral over A_0 for arbitrary ξ
- Another option: ignore ξ as a check and send

$$\xi \to 0\,. \tag{39.6}$$

- In this limit, the gauge condition becomes

$$\lim_{\xi\to 0} e^{-\frac{1}{2\xi}\int_x A_0^2} \to \prod_x \delta(A_0(x)) \tag{39.7}$$

- We can use the delta-function to perform the path-integral over A_0

Integrating out A_0

- The field strength tensor squared separates into

$$\frac{1}{2}F_{ab}F_{ab}=F_{0i}F_{0i}+\frac{1}{2}F_{ij}F_{ij}=\partial_0 A_i\partial_0 A_i+\frac{1}{2}F_{ij}F_{ij}-2\partial_i A_0\partial_0 A_i+(\partial_i A_0)^2 \tag{39.8}$$

- Using the δ function, we therefore have

$$Z_A=\int \mathcal{D}A_i e^{-\frac{1}{2}\int_x\left[\partial_0 A_i\partial_0 A_i+\frac{1}{2}F_{ij}F_{ij}\right]}\,, \tag{39.9}$$

where the measure $\mathcal{D}A_i$ is a reminder that only the spatial components of the gauge field remain to be integrated over

Fourier-Transforming Fields

- As in lecture 36, we now Fourier-transform the spatial gauge fields

$$A_i(x) = \frac{1}{\beta V} \sum_K e^{iK\cdot x} \tilde{A}_i(K)\,. \tag{39.10}$$

- Note that $K = (\omega_n, \mathbf{k})$ contains the time-like Matsubara frequencies, while we have already integrated out the time-like gauge field
- Using the Fourier-transform, we have

$$Z_A = \int \mathcal{D}\tilde{A}_i e^{-\frac{1}{2}\sum_K \tilde{A}_i(-K)\left[\omega_n^2\delta_{ij}+\mathbf{k}^2\delta_{ij}-k_i k_j\right]\tilde{A}_j(K)}\,, \tag{39.11}$$

The path integral

- Recall that in lecture 35, we had a problem with integrating the path integral

$$\int \mathcal{D}A_a e^{-\frac{1}{2}\sum_K \tilde{A}_a(-K)\left[K^2\delta_{ab}-K_aK_b\right]\tilde{A}_b(K)} \tag{39.12}$$

- The reason for this was that the matrix $\left[K^2\delta_{ab} - K_aK_b\right]$ had zero eigenvalues:

$$\left[K^2\delta_{ab} - K_aK_b\right] K_a = 0\,. \tag{39.13}$$

- Inspecting (39.11), we find that

$$Z_A = \prod_K \det{}^{-\frac{1}{2}}\left[K^2\delta_{ij} - k_ik_j\right] \tag{39.14}$$

- Do we have a problem similar to (39.13) again?

The path integral

- We can introduce two projectors

$$T_{ij} = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2}\,, \quad L_{ij} = \frac{k_i k_j}{\mathbf{k}^2}\,. \tag{39.15}$$

- In terms of these projectors, we have

$$\left[K^2\delta_{ij} - k_i k_j\right] = K^2 T_{ij} + \omega_n^2 L_{ij}\,. \tag{39.16}$$

- *Unlike* (39.13), this shows that (39.16) does not have a zero eigenvalue, so Z_A is well-defined

The path integral

- Since $\mathrm{Tr}\, T_{ij} = 2$, $\mathrm{Tr} L_{ij} = 1$, (39.16) has eigenvalue K^2 with degeneracy 2 and eigenvalue ω_n^2 with degeneracy one
- We find

$$Z_A = e^{-\frac{1}{2}\sum_K \ln\left[K^2\right]^2 - \frac{1}{2}\sum_K \ln\left[\omega_n^2\right]} \tag{39.17}$$

- The ghost contribution (39.4) is

$$Z_{\mathrm{ghost}} = e^{+\sum_K \ln \omega_n}\,. \tag{39.18}$$

- As was the case in lecture 36, we see that the ghosts cancel part of the gauge-field path integral
- In TAG, one can see that the ghosts cancel the *longitudinal* (L_{ij}) contribution, so that

$$Z = Z_A \times Z_{\mathrm{ghost}} = e^{-\frac{1}{2}\sum_K \ln\left[K^2\right]^2} \tag{39.19}$$

Only Transverse Photons Contribute

- The end result (39.19) matches the result (36.22) found in covariant gauges in lecture 36
- In TAG, we find that only the transverse photon contribution (the one arising from the eigenvalues of T_{ij}) contributes to the U(1) partition function
- While the cancellations between photons and ghosts work differently in different gauges, the end result is the same