The Casimir Effect

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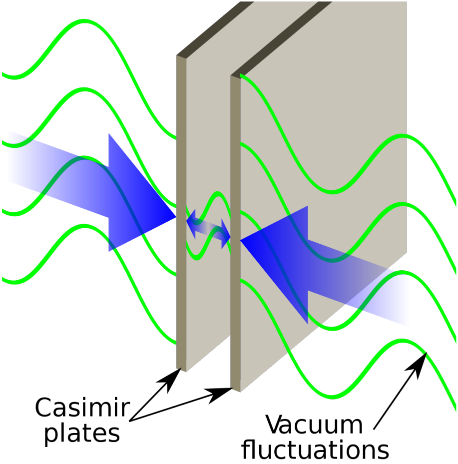
Review

- In lecture 39, we revisited the partition function in TAG, finding for the U(1) partition function

$$Z = e^{-\frac{1}{2}\sum_K \ln\left(K^2\right)^2} \tag{40.1}$$

- The power of 2 in the argument of the logarithm arose from the fact that Z only had contributions from transverse polarizations of the photon, while all others canceled out
- The sum in (40.1) is over all four momenta $K = (\omega_n, \mathbf{k})$
- In this lecture, we will use (40.1) to study the so-called Casimir effect

Casimir Effect



QFT in a finite volume

Because we need to consider QFT in a finite volume, there are a few differences wrt our usual treatment

- There are spatial boundary conditions, e.g. at $z = 0, L$
- If the plates at $z = 0, L$ are conducting, $E_{||}, B_{\perp}$ have to vanish there
- The pressure and free energy expressions differ. We now have

$$p = \frac{1}{\beta}\frac{\partial \ln Z}{\partial V}\,, \tag{40.2}$$

for the pressure, whereas the free energy is $\Omega = -\frac{\ln Z}{\beta V}$

Boundary conditions

Because we need to consider QFT in a finite volume, there are a few differences wrt our usual treatment

- If the plates at $z=0,L$ are conducting, $E_{||}, B_\perp$ have to vanish there
- This implies $A(t,x,y,z=0)=A(t,x,y,z=L)=0$
- We write $A(z)=\frac{1}{L}\sum_{k_z} e^{ik_z z}\tilde{A}(k_z)$ as before
- Since $A(z)$ is real, we have

$$A(z)=\frac{1}{L}\sum_{k_z}\left[\cos(k_z z)\mathrm{Re}\tilde{A}(k_z)-\sin(k_z z)\mathrm{Im}\tilde{A}(k_z)\right]\,. \tag{40.3}$$

- $A(z=0)=0$ implies $\mathrm{Re}\tilde{A}(k_z)=0$
- The boundary condition at $z=L$ then implies

$$k_z=\frac{m\pi}{L}\,,\quad m\in Z \tag{40.4}$$

Path Integrals – “Normal” BCs

- When performing the path integral, we are integrating exponentials of quadratic forms, e.g. in one dimension

$$\int \mathcal{D}\phi e^{-\frac{1}{2}\int_x m^2\phi^2(x)}\,. \tag{40.5}$$

- Fourier-transforming the fields (without non-trivial BCs) we have

$$\int \mathcal{D}\phi e^{-\frac{1}{2}\sum_k m^2\phi(k)\phi^*(k)} = \int \mathcal{D}a_k\mathcal{D}b_k e^{-\frac{m^2}{2}\sum_k\left[a_k^2+b_k^2\right]}\,, \tag{40.6}$$

where $\phi(k) = a_k + ib_k$

- For open boundary conditions, the relation $\phi^*(k) = \phi(-k)$ leads to the restriction

$$\int \mathcal{D}\phi e^{-\frac{1}{2}\sum_k m^2\phi(k)\phi^*(k)} = \int \mathcal{D}a_k\mathcal{D}b_k e^{-m^2\sum_{k=0}^{\infty}\left[a_k^2+b_k^2\right]}\,. \tag{40.7}$$

Path Integrals – "Normal" BCs

- Each of the integrals a_k, b_k gives

$$\int \mathcal{D}a_k e^{-m^2\sum_{k=0}^{\infty} a_k^2} = \prod_{k=0}^{\infty}\sqrt{\frac{\pi}{m^2}} \tag{40.8}$$

- Therefore

$$\begin{aligned}\int \mathcal{D}\phi e^{-\frac{1}{2}\sum_k m^2\phi(k)\phi^*(k)} &= \prod_{k=0}^{\infty}\left(\sqrt{\frac{\pi}{m^2}}\right)^2, \\ &= \prod_{k=-\infty}^{\infty}\left(\sqrt{\frac{\pi}{m^2}}\right) = e^{-\frac{1}{2}\sum_k \ln\frac{\pi}{m^2}}.\end{aligned} \tag{40.9}$$

Path Integrals – Conducting Plates BCs

- In the case of conducting plates, the imaginary part vanishes, so instead of

$$\int \mathcal{D}a_k \mathcal{D}b_k e^{-m^2 \sum_{k=0}^{\infty}\left[a_k^2+b_k^2\right]} = e^{-\frac{1}{2}\sum_k \ln\frac{\pi}{m^2}} \tag{40.10}$$

  we have

$$\int \mathcal{D}a_k e^{-m^2 \sum_{k=0}^{\infty}\left[a_k^2\right]} = e^{-\frac{1}{4}\sum_k \ln\frac{\pi}{m^2}}\,. \tag{40.11}$$

- Therefore, we need to adjust the formula (40.1) for the partition function to

$$Z = e^{-\frac{1}{4}\sum_K \ln\left(K^2\right)^2}\,. \tag{40.12}$$

Partition function

- The partition function for the conducting plates becomes from (40.12)

$$\ln Z = -\frac{1}{2}\sum_{\omega_n}\sum_{k_\perp}\sum_{k_z}\ln\left(\omega_n^2 + k_\perp^2 + k_z^2\right)\,, \qquad (40.13)$$

  where $k_\perp^2 = k_x^2 + k_y^2$
- Assuming the conducting plates to be parallel and infinitely large, $k_\perp$ becomes continuous in the large volume limit
- We can perform the sum over the Matsubara frequencies as in lecture 6, finding

$$\ln Z = -\sum_{k_\perp}\sum_{k_z}\left[\frac{\beta\sqrt{k_\perp^2 + k_z^2}}{2} + \ln\left(1 - e^{-\beta\sqrt{k_\perp^2+k_z^2}}\right)\right]. \qquad (40.14)$$

Partition function

- Concentrating on the zero temperature limit $\beta \to \infty$, we have

$$\ln Z = -\frac{\beta}{2} \sum_{k_\perp} \sum_{k_z} \sqrt{k_\perp^2 + k_z^2} \tag{40.15}$$

- Since k_z values are discrete, we have

$$\ln Z = -\frac{\beta}{2} \sum_{k_\perp} \sum_{m=-\infty}^{\infty} \sqrt{k_\perp^2 + \left(\frac{m\pi}{L}\right)^2} \tag{40.16}$$

- If we make the perpendicular directions very large, this becomes

$$\ln Z = -\frac{\beta V_\perp}{2} \int \frac{d^2 k_\perp}{(2\pi)^2} \sum_{m=-\infty}^{\infty} \sqrt{k_\perp^2 + \left(\frac{m\pi}{L}\right)^2} \tag{40.17}$$

Partition function

- The integral over $k_\perp$ can be done using the function Φ defined in lecture 10:

$$\Phi(m,D,A) \equiv \int \frac{d^D k}{(2\pi)^D}\left(k^2+m^2\right)^{-A} = \frac{1}{(4\pi)^{\frac{D}{2}}}\frac{\Gamma\left(A-\frac{D}{2}\right)}{\Gamma(A)}\left(m^2\right)^{-A+\frac{D}{2}}\,. \tag{40.18}$$

- We find with $D=2-2\varepsilon$, $A=-\frac{1}{2}$:

$$\ln Z = +\frac{\beta V_\perp}{2}\sum_{m=-\infty}^{\infty}\frac{1}{6\pi}\left(\frac{m^2\pi^2}{L^2}\right)^{\frac{3}{2}} = \beta V_\perp \sum_{m=1}^{\infty}\frac{1}{6\pi}\left(\frac{m^2\pi^2}{L^2}\right)^{\frac{3}{2}}\,. \tag{40.19}$$

Partition function

- The Riemann ζ function is defined as

$$\zeta(s) \equiv \sum_{m=1}^{\infty} \frac{1}{m^s} \,. \tag{40.20}$$

- Using the ζ-function, we have

$$\ln Z = \beta V_\perp \frac{\pi^3}{6\pi L^3} \sum_{m=1}^{\infty} m^3 = \frac{\beta V_\perp \pi^2}{6 L^3} \zeta(-3)\,. \tag{40.21}$$

- Since dim-reg and ζ-function regularization are the same, we can use

$$\zeta(-3) = \frac{1}{120} \,. \tag{40.22}$$

Partition function

- This leads to

$$\ln Z = \frac{\beta V_\perp \pi^2}{720} L^{-3}\,. \tag{40.23}$$

- As a result, the Casimir pressure is

$$p = \frac{1}{\beta}\frac{\partial \ln Z}{\partial V} = \frac{1}{\beta V_\perp}\frac{\partial \ln Z}{\partial L} = \frac{\pi^2}{720}\frac{(-3)}{L^4}\,. \tag{40.24}$$

- We find the result

$$p = -\frac{\pi^2}{240 L^4}\,. \tag{40.25}$$

Casimir Force

- The pressure is *negative*
- The pressure is force per unit area
- For a boundary plate area of $V_\perp$, this implies the force per unit area

$$\frac{F}{V_\perp} = p = -\frac{\pi^2}{240 L^4} \,. \tag{40.26}$$

- The force on the boundary plates is attractive
- We can convert to SI units:

$$\frac{F}{V_\perp} = -\frac{\pi^2}{240 L^4}\hbar c \simeq -1.2 \times 10^{-27} N \frac{m^2}{L^4} \,. \tag{40.27}$$

Casimir Force

- The Casimir force is a *prediction* from QFT
- It has been experimentally verified with high precision experiments, e.g. in [Mohideen and Roy, PRL81, 1998]

Casimir Force