Magnetic Moment of the Electron

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Fall 2020

Review

- In lecture 35, we discussed QED
- One of the precision tests of QED is the *anomalous magnetic moment* of the electron, also known as $g-2$
- In this lecture, we set the stage for calculating the anomalous magnetic moment, by first calculating g
- In future lectures, we will go on to discuss calculating $g-2$ in QED

QED

- From lecture 35, we have the QED partition function given by

$$Z = \int \mathcal{D}A\mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}\bar{c}\mathcal{D}c e^{-S_{\rm matter}-S_{\rm gauge}-S_{\rm gf}-S_{\rm ghosts}}\,. \tag{41.1}$$

- Here $S_{\rm matter}$, $S_{\rm gauge}$, $S_{\rm gf}$, $S_{\rm ghosts}$ are the matter, gauge, gauge-fixing and ghost parts of the action
- For an electron, the matter fields are Dirac fermions
- Requiring the action to be invariant under gauge transformations

$$S_{\rm matter} = \int_x \bar{\psi}\left(\not{D} + m\right)\psi\,, \tag{41.2}$$

where D_μ is the covariant derivative and $\not{A} \equiv A_\mu \gamma_E^\mu$

Gauge Invariance

Under local U(1) gauge transformations

$$\psi(x) \to e^{i\alpha(x)}\psi(x)\,, \tag{41.3}$$

the matter part of the action is invariant if

- the covariant derivative is given by

$$D_\mu = \partial_\mu + iA_\mu(x)\,, \tag{41.4}$$

- and the gauge field transforms as

$$A_\mu(x) \to A_\mu(x) - \partial_\mu\alpha(x)\,. \tag{41.5}$$

- In addition to the matter part, also the gauge-part is invariant if

$$S_{\rm gauge} = \frac{1}{4e^2}\int_x F_{\mu\nu}F_{\mu\nu}\,,\quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\,. \tag{41.6}$$

Field transform

- The normalization of the gauge field is arbitrary so far
- We can change the normalization of the gauge field by a constant, e.g.

$$A_\mu \to e A_\mu \,. \tag{41.7}$$

- This will change the relevant parts of the action as

$$S_{\rm matter} + S_{\rm gauge} = \int_x \left[\bar\psi \left(\not\partial + m + ie \not{A} \right) \psi + \frac{1}{4} F_{\mu\nu} F_{\mu\nu} \right] \tag{41.8}$$

- We will use this normalization for the action in the following

Hamiltonian

- We calculated the Hamiltonian for a Dirac fermion in lecture 32:

$$H = \int d^3x \bar{\psi} \left(-i\gamma^i \partial_i + m\right) \psi \tag{41.9}$$

- Note that here γ^i denote the *Minkowski* gamma matrices
- Using the relations (32.12) to Euclidean gamma matrices γ_μ^E we have

$$H = \int d^3x \bar{\psi} \left(\gamma_i^E \partial_i + m\right) \psi \tag{41.10}$$

- It's easy to see that this corresponds to part of the matter action $S_{\rm matter}$ in (41.8)

Hamiltonian for QED

- The classical Hamiltonian (41.10) is for a single Dirac fermion
- For QED, there is a coupling between fermion and gauge field, according to $S_{\rm matter}$, cf. (41.8)
- For QED, the relevant classical Hamiltonian contribution is

$$\Delta H = ie\int d^3x \bar\psi \not{A}\psi = ie\int d^3x \bar\psi \gamma_\mu^E \psi A_\mu \,. \tag{41.11}$$

- For simplicity, let's consider TAG $A_0 = 0$ so that

$$\Delta H = ie\int d^3x \bar\psi \gamma_i^E \psi A_i \,. \tag{41.12}$$

- Let us rewrite this Hamiltonian contribution in the limit of small A_μ

Gordon Identity

- If the gauge field amplitude is small, then the classical fermions fulfill the Dirac equation,

$$\left(i\gamma^\mu \partial_\mu - m\right)\psi = 0\,. \tag{41.13}$$

- For time-independent fields $\psi = \psi(\vec{x})$ and using Eulidean γ matrices, we thus have

$$m\psi = -\gamma_E^i \partial_i \psi \tag{41.14}$$

- Taking the Hermitian conjugate, and using $\gamma_E^{i\dagger} = \gamma_E^i$ we have

$$m\bar{\psi} = \left(\partial_i \bar{\psi}\right)\gamma_E^i\,. \tag{41.15}$$

Gordon Identity

- Multiplying these equations by $\bar{\psi}\gamma_E^j$ and $\gamma_E^j\psi$ from left and right, respectively, gives

$$
\begin{aligned}
m\bar{\psi}\gamma_E^j\psi &= -\bar{\psi}\gamma_E^j\gamma_E^i\partial_i\psi\,,\\
m\bar{\psi}\gamma_E^j\psi &= \left(\partial_i\bar{\psi}\right)\gamma_E^i\gamma_E^j\psi\,.
\end{aligned}
\tag{41.16}
$$

- Summing these then leads to the *Gordon identity*

$$
\bar{\psi}\gamma_E^j\psi = \frac{1}{2m}\left[\left(\partial_i\bar{\psi}\right)\gamma_E^i\gamma_E^j\psi - \bar{\psi}\gamma_E^j\gamma_E^i\partial_i\psi\right]
\tag{41.17}
$$

- The identity can be simplified by using $\sigma^{ij} = \frac{i}{2}\left[\gamma_E^i, \gamma_E^j\right]$:

$$
\gamma_E^i\gamma_E^j = \frac{1}{2}\left\{\gamma_E^i, \gamma_E^j\right\} + \frac{1}{2}\left[\gamma_E^i, \gamma_E^j\right] = \delta^{ij} - i\sigma^{ij}\,.
\tag{41.18}
$$

Hamiltonian Contribution

- We get for the Gordon identity:

$$\bar{\psi}\gamma_E^j\psi = \frac{1}{2m}\left[\left(\partial_j\bar{\psi}\right)\psi - \bar{\psi}\partial_j\psi - i\partial_i\left(\bar{\psi}\sigma^{ij}\psi\right)\right] \tag{41.19}$$

- As a consequence, the Hamiltonian contribution (41.12) becomes

$$\Delta H = \frac{ie}{2m}\int d^3x\left[\left(\partial_j\bar{\psi}\right)\psi - \bar{\psi}\partial_j\psi\right]A_j + \frac{e}{2m}\int d^3x\partial_i\left(\bar{\psi}\sigma^{ij}\psi\right)A_j\,. \tag{41.20}$$

- For now, we are not interested in the first part. Instead, we focus on

$$\Delta H^{\mathrm{spin-orbit}} = -\frac{e}{2m}\int d^3x\bar{\psi}\sigma^{ij}\psi\frac{1}{2}F_{ij}\,. \tag{41.21}$$

Hamiltonian Contribution

- We can consider $\Delta H^{\text{spin}-\text{orbit}}$ for an *external constant* magnetic field $B_3 = F_{12}$
- In this case

$$\Delta H^{\text{spin}-\text{orbit}} = -\mu_B \int d^3x \bar{\psi}\sigma^{12}\psi B_3 \,. \tag{41.22}$$

where we have introduced the *Bohr-magneton*

$$\mu_B = \frac{e}{2m} \,. \tag{41.23}$$

- Now

$$\sigma^{12} = \frac{i}{2}\left[\gamma_E^1, \gamma_E^2\right] = -\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \,. \tag{41.24}$$

Spin-Orbit Interaction

- For a fermion, being a spin-$\frac{1}{2}$ particle the spin operator is

$$S_3 = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \tag{41.25}$$

- Using the spin-operator, the Hamiltonian contribution becomes

$$\Delta H^{\rm spin-orbit} = 2\mu_B \int d^3x \bar{\psi}\vec{S}\cdot\vec{B}\psi\,. \tag{41.26}$$

- The non-trivial pre-factor 2 is the magnetic moment of the Dirac particle
- It is customarily denoted as g and called "g-factor"

From classical to quantum

- While our treatment was relativistic, and contained fermions, it nevertheless did not contain any true quantum field theory effects
- For this reason, the result $g = 2$ is a classical approximation to the actual g-factor
- The *deviation* from $g = 2$ is called the *anomalous contribution* to the magnetic moment (though there is nothing anomalous about it)
- We will consider one-loop QFT approximations for $g - 2$ in the next lectures