g-2

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Review

- In lecture 41, we discussed the magnetic moment of the electron
- Because of the half-integer spin, there is a correction factor called g
- In lecture 41, we found that $g = 2$ in the classical approximation (Dirac equation)
- In this lecture, we will set up QFT corrections to $g - 2$

Vertex Corrections

- Recall from lecture 41, that the QED Hamiltonian contains the relevant term

$$\Delta H = ie \int d^3 x \bar{\psi} \not{A} \psi \,. \tag{42.1}$$

- That is the same term that appears in the QED matter action:

$$S_{\rm matter} = \int_x \bar{\psi} \left(\not{\partial} + ie \not{A} \right) \psi \,. \tag{42.2}$$

- QFT corrections will change the effective fermion-gauge field coupling
- We can calculate these as “loop corrections” in perturbation theory for $e \ll 1$

Vertex Corrections

- For the classical theory, the fermion-photon vertex is simply

$$ie\bar{\psi}\gamma_\mu^E A_\mu \psi \tag{42.3}$$

- In QFT, we instead get the *expectation value* of the corresponding vertex operator:

$$\Gamma_\mu(x_1, x_2, x_3) = \langle \bar{\psi}(x_1) A_\mu(x_2) \psi(x_3) \rangle_{\rm full}\,. \tag{42.4}$$

- Here the subscript $\rm full$ refers to taking the expectation value in the full QFT (i.e. with interaction terms), not in the free theory (where $e = 0$)
- To lowest non-trivial order in perturbation theory, we have

$$\Gamma_\mu^{(1)}(x_1, x_2, x_3) = -\langle S_{\rm matter} \bar{\psi}(x_1) A_\mu(x_2) \psi(x_3) \rangle\,. \tag{42.5}$$

Vertex Corrections

- As in lecture 20, using (42.2) for $S_{\rm matter}$, we have

$$\Gamma^{(1)}_\mu(x_1,x_2,x_3) = -ie\gamma^E_\nu \int_x \langle \bar\psi(x) A_\nu(x)\psi(x)\bar\psi(x_1)A_\mu(x_2)\psi(x_3)\rangle \,. \tag{42.6}$$

- The expression under the integral reduces to two free fermion propagators $\langle \psi\bar\psi\rangle$, and a free photon propagator $\langle A_\mu A_\nu\rangle$
- The integral over x enforces momentum-conservation
- We can define a *connected, amputated* vertex function by disregarding these elements (cf. lecture 20)

Amputated Vertex

- The amputated connected vertex function to leading order in perturbation theory therefore is

$$\Gamma_\mu^{(1),\mathrm{conn.,amp.}} = ie\gamma_\mu^E \tag{42.7}$$

- To leading order in perturbation theory, the relevant term in the QED Hamiltonian (42.1) is thus

$$\Delta H^{(1)} = \int d^3x \bar{\psi} A_\mu \Gamma_\mu^{(1),\mathrm{conn.,amp.}} \psi\,. \tag{42.8}$$

- Higher orders in perturbationt theory work the same way, so that in QFT

$$\Delta H = \int d^3x \bar{\psi} A_\mu \Gamma_\mu^{\mathrm{conn.,amp.}} \psi\,. \tag{42.9}$$

Amputated Vertex

- In order to calculate QED corrections to the g-factor, we need to calculate corrections to the fermion-photon vertex
- It is advantageous to work in Fourier space, where $\tilde{\Gamma}_\mu(p, p')$ only depends on two external momenta (because of momentum-conservation)
- It will turn out that we can decompose

$$\Gamma_\mu^{\rm conn.,amp.}(p, p') = ie\gamma_\mu^E F_1(q^2) + \frac{ie}{2m}\sigma_{\mu\nu} q_\nu F_2(q^2)\,, \qquad (42.10)$$

  where p, p' are chosen as the 4-momenta of the incoming and outgoing electrons, and $q = p' - p$.
- Here F_1, F_2 are so-called *form factors*

Form Factors and g-2

- From first-order perturbation theory, we have

$$F_1(q^2) = 1\,, \quad F_2(q^2) = 0\,. \tag{42.11}$$

- Using the Gordon identity (41.19) in momentum space

$$\bar{\psi}(p')\gamma_\mu^E\psi(p) = -\frac{1}{2m}\bar{\psi}(p')\left[i(p'_\mu + p_\mu) - \sigma_{\mu\nu}q_\nu\right]\psi(p) \tag{42.12}$$

we can replace γ_E in the connected vertex

- One gets

$$\Gamma_\mu^{\rm conn.,amp.}(p,p') = \frac{e}{2m}\left(p_\mu + p'_\mu\right)F_1(q^2) + \frac{ie}{2m}\sigma_{\mu\nu}q_\nu\left(F_1(q^2)+F_2(q^2)\right)\,, \tag{42.13}$$

Form Factors and g-2

- The first part of the vertex does not contribute to the magnetic moment
- The second part contributes when relating $q_\nu A_\mu(q)$ to the magnetic field, so there is a relation between g and the form factors F_1, F_2
- When calculating the energy change from the spin-orbit coupling, we evaluate $\langle \Delta H \rangle$
- Since the external B-field is assumed constant, we have $q = 0$
- Therefore,

$$g = 2\left(F_1(0) + F_2(0)\right) . \qquad (42.14)$$

Renormalization

- The form factors F_1, F_2 will in general be divergent
- In order to renormalize the theory, we will need to introduce counter-terms for the parameters in the Lagrangian, such as the electron charge e:

$$e \to e_{\rm phys} + \delta e\,. \tag{42.15}$$

- It is possible to choose a scheme ("on-shell" or "OS") such that

$$\Gamma_\mu^{\rm conn.,amp.}(q=0) = ie\gamma_\mu^E F_1(0) \equiv ie_{\rm phys}\gamma_\mu^E\,, \tag{42.16}$$

such that $F_1(0)=1$ (exact) in this scheme

g-2

- In the OS renormalization scheme where $F_1(0) = 1$ we have

$$g = 2\left(1 + F_2^{\rm OS}(0)\right) \tag{42.17}$$

such that

$$g - 2 = 2F_2^{\rm OS}(0)\,. \tag{42.18}$$

- We proceed to calculate $F_2^{\rm OS}(0)$ in the next lecture