g-2, part II

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Review

- In lecture 42, we discussed that g-2 is related to the connected and amputated photon-electron vertex $\Gamma_\mu^{\rm conn.,amp.}$
- In this lecture, we start the calculation of $\Gamma_\mu^{\rm conn.,amp.}$
- We will take this as a practical opportunity to discuss many details of QED calculations

Setting up the Calculation

- In lecture 41, we calculated the photon-electron vertex in QED to leading order in perturbation theory
- Since the interaction term in $S_{\rm matter} \propto ie \int_x \bar\psi \not{A} \psi$, the second-order contribution to Γ_μ vanishes identically (odd number of A_μ's)
- Therefore, the next non-trivial perturbative correction is third order

Third order vertex

- Expanding $e^{-S_{\rm matter}}$ to third order in the coupling e we have

$$\Gamma^{(3)}_\mu(x_1,x_2,x_3)=\frac{(-ie)^3}{3!}\int_{x,y,z}\langle\bar\psi_x \not{A}_x\psi_x\bar\psi_y\not{A}_y\psi_y\bar\psi_z\not{A}_z\psi_z\bar\psi_1 A_{2,\mu}\psi_3\rangle\,. \tag{43.1}$$

- For notational brevity, the space-time coordinate has been written as a subscript, and ψ_1 is meant to represent $\psi(x_1)$
- The expectation value $\langle\cdot\rangle$ is wrt the free theory, which is Gaussian
- For this reason, the expectation value decomposes into the familiar Wick contractions

Wick contractions

- First we start contracting the fermions, using the basic fermion propagator
$$S_{x,y}^{\alpha,\beta} = \langle \psi_x^\alpha \bar{\psi}_y^\beta \rangle \,, \tag{43.2}$$
where we have made the spin indices α, β explicit.
- Starting with $\bar{\psi}_1$, we have the option of contracting it with ψ_x, ψ_y, ψ_z; this corresponds to three ways of relabeling coordinates, which will result in the same integral
- We choose contraction with ψ_z and multiply this contribution by three:

$$\Gamma_\mu^{(3)}(x_1,x_2,x_3) = \frac{(-ie)^3}{2!}\int_{x,y,z} \langle \bar{\psi}_x \not{A}_x \psi_x \bar{\psi}_y \not{A}_y \psi_y \bar{\psi}_z^\epsilon A_{z,\delta} \gamma_\delta^{\epsilon,\epsilon'} A_{2,\mu}\psi_3 \rangle S_{z,1}^{\epsilon',\cdot} \,. \tag{43.3}$$

Wick contractions

- Next we pick ψ_3
- We can connect it to $\bar{\psi}_x$ or $\bar{\psi}_y$, which again gives the same result up to relabeling of coordinates
- Choosing $\bar{\psi}_y$, we have

$$\Gamma^{(3)}_\mu = -ie^3 \int_{x,y,z} \langle \bar{\psi}_x \not{A}_x \psi_x A_{y,\sigma} \gamma_\sigma^{\beta,\beta'} \psi_y^{\beta'} \bar{\psi}_z^{\epsilon} A_{z,\delta} \gamma_\delta^{\epsilon,\epsilon'} A_{2,\mu} \rangle S_{3,y}^{\cdot,\beta} S_{z,1}^{\epsilon',\cdot} \,. \tag{43.4}$$

- Note the minus sign that arose from pulling $\bar{\psi}_z$ through an *odd* number of spinors, which are anti-commuting

Wick contractions

- Next, we pick $\bar{\psi}_z$
- We can connect it to ψ_y or ψ_x, but only connecting it to ψ_x will give us a *connected* contribution
- Choosing this, we have

$$\Gamma^{(3)}_\mu = +ie^3 \int_{x,y,z} \langle \bar{\psi}^\alpha_x A_{x,\rho} \gamma_\rho^{\alpha,\alpha'} A_{y,\sigma} \gamma_\sigma^{\beta,\beta'} \psi_y^{\beta'} A_{z,\delta} \gamma_\delta^{\epsilon,\epsilon'} A_{2,\mu} \rangle S_{x,z}^{\alpha',\epsilon} S_{3,y}^{\cdot,\beta} S_{z,1}^{\epsilon',\cdot} \,. \tag{43.5}$$

Wick contractions

- Finally, rearranging the remaining $\psi_y, \bar{\psi}_x$ such that they appear in the order $\psi_y \bar{\psi}_x$ we have

$$\Gamma_\mu^{(3)} = -ie^3 \int_{x,y,z} \langle A_{x,\rho} \gamma_\rho^{\alpha,\alpha'} A_{y,\sigma} \gamma_\sigma^{\beta,\beta'} A_{z,\delta} \gamma_\delta^{\epsilon,\epsilon'} A_{2,\mu} \rangle S_{y,x}^{\beta',\alpha} S_{x,z}^{\alpha',\epsilon} S_{3,y}^{\cdot,\beta} S_{z,1}^{\epsilon',\cdot} . \tag{43.6}$$

- We now need to contract the photons
- Starting with A_2, we have the option of connecting it to A_x or A_y
- Connecting it to A_y gives a connected contribution — but it's "trivial" because it just corresponds to $\Gamma_\mu^{(1)}$ with a second-order propagator

Wick contractions

- Connecting therefore A_2, A_x as

$$\langle A_{2,\mu} A_{x,\rho} \rangle = G^{\mu\rho}_{2,x} \tag{43.7}$$

we have

$$\Gamma^{(3)}_\mu = -ie^3 \int_{x,y,z} G^{\sigma\delta}_{y,z} G^{\mu\rho}_{2,x} \gamma^{\alpha,\alpha'}_\rho \gamma^{\beta,\beta'}_\sigma \gamma^{\epsilon,\epsilon'}_\delta S^{\beta',\alpha}_{y,x} S^{\alpha',\epsilon}_{x,z} S^{\cdot,\beta}_{3,y} S^{\epsilon',\cdot}_{z,1} \,. \tag{43.8}$$

- Writing the spin-indices in-order so that they correspond to matrix multiplications gives

$$\Gamma^{(3)}_\mu(x_1,x_2,x_3) = -ie^3 \int_{x,y,z} G^{\sigma\delta}_{y,z} G^{\mu\rho}_{2,x} S_{3,y}\gamma_\sigma S_{y,x}\gamma_\rho S_{x,z}\gamma_\delta S_{z,1} \,. \tag{43.9}$$

Fourier Transformations

- Next, we Fourier-transform all propagators, writing e.g.

$$S_{3,y} = \int_K e^{iK\cdot(x_3-y)} \tag{43.10}$$

- We get

$$\begin{aligned}\Gamma^{(3)}_\mu(x_1,x_2,x_3) = -ie^3 \int_{x,y,z}\int_{K_1,K_2,K_3,K_4,K_5,K_6} & G_1^{\sigma\delta}G_2^{\mu\rho}S_3\gamma_\sigma S_4\gamma_\rho S_5\gamma_\delta S_6\\ &\times e^{ix\cdot(K_2-K_4+K_5)+iy\cdot(-K_1-K_3+K_4)+iz(K_1-K_5+K_6)}\\ &\times e^{-ix_1\cdot K_6-ix_2\cdot K_2+ix_3\cdot K_3}\,,\end{aligned} \tag{43.11}$$

where S_1 now denotes $S(K_1)$

Fourier Transformations

- Integrating over x, y, z gives three delta functions:

$$\delta(K_2 - K_4 + K_5)\delta(K_1 + K_3 - K_4)\delta(K_1 - K_5 + K_6) \tag{43.12}$$

- Doing an inverse Fourier transform on x_1, x_2, x_3 as

$$e^{+ix_1\cdot P - ix_3\cdot P' - iQ\cdot x_2} \tag{43.13}$$

  gives three additional δ-functions
- We use the delta functions to do the momentum integrations K_2, K_3, K_4, K_5, K_6
- Relabeling $K_1 \equiv K$ we have

$$\Gamma^{(3)}_\mu(P, P') = -ie^3 \int_K G_K^{\rho\sigma} G_Q^{\mu\rho} S_{P'}\gamma_\sigma S_{K+P'}\gamma_\rho S_{K+P}\gamma_\delta S_P \delta(Q - P' + P) \tag{43.14}$$

Amputated Vertex

- We recognize $\delta(Q-P'+P)$ as the constraint of momentum conservation
- We also recognize $G_Q^{\mu\rho}$ as propagator for the photon of momentum Q, and $S_P, S_{P'}$ as the external fermion propagators
- This is the structure anticipated in lecture 42
- We therefore get the *amputated* vertex as

$$\Gamma_\mu^{(3),\text{conn.,amp.}}(P,P') = -ie^3 \int_K G_K^{\rho\sigma}\gamma_\sigma S_{K+P'}\gamma_\mu S_{K+P}\gamma_\delta \,. \qquad (43.15)$$

- We will evaluate the connected vertex in the next lectures