Fermion Propagator

paul.romatschke@colorado.edu

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Review

- In lecture 43, we derived the amputated photon-electron vertex in QED
- A part of the result was the fermion propagator

$$S(x) \equiv \langle \psi(x) \bar{\psi}(0) \rangle \tag{44.1}$$

  in the free theory
- We discussed the fermion partition function in lecture 33, which we will use as a starting point

Fermion Partition Function

- The fermion partition function was derived in lecture 32
- We found

$$Z_F = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-\int_x \bar{\psi}\left[\gamma_\mu^E \partial_\mu + m\right]\psi} . \qquad (44.2)$$

- A key difference wrt bosonic theories is that fermions obey anti-periodic boundary conditions, e.g. $\psi(\tau=\beta) = -\psi(\tau=0)$
- Performing a Fourier transform as in lecture 33 we have

$$\psi(\tau, \mathbf{x}) = \frac{T}{V}\sum_K e^{iK\cdot x}\tilde{\psi}(K) . \qquad (44.3)$$

- The fermionic Matsubara frequencies are $K_0 = \pi(2n+1)T$

Propagator

- The free fermion propagator is given by

$$S(x) = \frac{\int \mathcal{D}\bar{\psi}\mathcal{D}\psi\psi(x)\bar{\psi}(0)e^{-\int_x \bar{\psi}\left[\gamma_\mu^E \partial_\mu + m\right]\psi}}{Z} \tag{44.4}$$

- Fourier-transforming this gives

$$\tilde{S}(P) = \frac{T}{V}\sum_Q \frac{\int \mathcal{D}\bar{\tilde{\psi}}\mathcal{D}\tilde{\psi}\tilde{\psi}(P)\bar{\tilde{\psi}}(Q)e^{-\frac{T}{V}\sum_K \bar{\tilde{\psi}}\left[i\gamma_\mu^E K_\mu + m\right]\tilde{\psi}}}{Z} \tag{44.5}$$

Grassmann integration

- Using the rules of Grassmann integration, we have

$$\frac{\int dc^* dc c c^* e^{-c^* a c}}{\int dc^* dc e^{-c^* a c}} = \frac{\int dc^* dc c c^*}{\int dc^* dc(-c^* a c)} = \frac{1}{a}\,. \tag{44.6}$$

- For the multi-component case, this is

$$\frac{\int \left[\prod_i dc_i^* dc_i\right] c_k c_l^* e^{-c_i^* M_{ij} c_j}}{\int \left[\prod_i dc_i^* dc_i\right] e^{-c_i^* M_{ij} c_j}} = \frac{1}{M_{kl}} \tag{44.7}$$

Free Propagator

- The relevant integral for the fermion propagator is then

$$\frac{\int \mathcal{D}\bar{\tilde{\psi}}\mathcal{D}\tilde{\psi}\tilde{\psi}(P)\bar{\tilde{\psi}}(Q)e^{-\frac{T}{V}\sum_K \bar{\tilde{\psi}}[i\gamma_\mu^E K_\mu + m]\tilde{\psi}}}{Z} = \frac{V}{T}\frac{1}{i\not{P}+m}\delta_{P,Q}\,. \quad (44.8)$$

- The free fermion propagator therefore becomes

$$\tilde{S}(P) = \frac{1}{i\not{P}+m}\,. \quad (44.9)$$

Rewriting the propagator

- We can rewrite the propagator as

$$\tilde{S}(P) = \frac{1}{i\not{P} + m} = \frac{1}{\left(i\not{P} + m\right)\left(-i\not{P} + m\right)}\left(-i\not{P} + m\right) \qquad (44.10)$$

- In addition, we have

$$\begin{aligned} \left(i\not{P} + m\right)\left(-i\not{P} + m\right) &= \not{P}\not{P} + m^2 \\ &= \frac{1}{2} P_\mu P_\nu \left\{\gamma_\mu^E, \gamma_\nu^E\right\} + m^2 \\ &= P^2 + m^2 \qquad (44.11) \end{aligned}$$

- We can use this to rewrite

$$\tilde{S}(P) = \frac{-i\not{P} + m}{P^2 + m^2}\,. \qquad (44.12)$$