g-2, part III

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Fall 2020

Review

- In lecture 42, we discussed that g-2 is related to the connected and amputated photon-electron vertex $\Gamma_\mu^{\text{conn.,amp.}}$
- In lecture 43, we derived the result

$$\Gamma_\mu^{(3),\text{conn.,amp.}}(P,P') = -ie^3 \int_K G_K^{\rho\sigma}\gamma_\sigma S_{K+P'}\gamma_\mu S_{K+P}\gamma_\delta \qquad (46.1)$$

- In this lecture, we will start with the evaluation of the vertex

Propagators

- In Eq. (46.1), $G^{\rho\sigma}$ refers to the photon propagator
- We discussed the photon propagator in lecture 45
- To make the calculation of Γ as simple as possible, we choose Feynman gauge $\xi = 1$ so that

$$G_K^{\rho\sigma} = \frac{\delta^{\rho\sigma}}{K^2} \,. \tag{46.2}$$

- Also, we discussed the fermion propagator S_K in lecture 44
- It is given by

$$S_K = \frac{-i\not{K} + m}{K^2 + m^2} \,. \tag{46.3}$$

Vertex

- Using the explicit propagators in (46.1), one finds for the vertex

$$\Gamma_\mu^{(3),\text{conn.,amp.}}(P,P') = -ie^3 \int_K \frac{N_\mu}{K^2\left[(K+P')^2+m^2\right]\left[(K+P)^2+m^2\right]}\,, \tag{46.4}$$

where the numerator is given by

$$N_\mu = \gamma_\sigma \left[i\left(\not{K}+\not{P}'\right)-m\right]\gamma_\mu\left[i\left(\not{K}+\not{P}\right)-m\right]\,. \tag{46.5}$$

- We want to simplify the integral by combining the terms in the denominator
- This can be done using Feynman parameters

Feynman Parameters

- Feynman observed that for all $A, B \neq 0$

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{\left(Ax + (1-x)B\right)^2}\,. \tag{46.6}$$

- Here x is called a Feynman parameter
- The above identity can be generalized to

$$\frac{1}{A_1 A_2 \ldots A_n} = (n-1)! \int_0^1 dx_1 dx_2 \ldots dx_n \frac{\delta\left(1 - x_1 - x_2 - \ldots - x_n\right)}{\left(x_1 A_1 + x_2 A_2 + \ldots + x_n A_n\right)^n} \tag{46.7}$$

Feynman Parameters

- In our case, we are going to use

$$\frac{1}{K^2\left[(K+P')^2+m^2\right]\left[(K+P)^2+m^2\right]}=\int_0^1 dx_1 dx_2 \frac{2}{D^3}\,, \qquad (46.8)$$

where

$$\begin{aligned} D &= K^2(1-x_1-x_2)+x_2((K+P')^2+m^2)+x_1((K+P)^2+m^2) \\ &= K^2+2x_1 K\cdot P+2x_2 K\cdot P'+x_1 P^2+x_2 P'^2+m^2(x_1+x_2) \end{aligned}$$

- We have for the vertex

$$\Gamma_\mu^{(3),\mathrm{conn.,amp.}}(P,P')=-2ie^3\int_0^1 dx_1 dx_2 \int_K \frac{N_\mu}{D^3}\,. \qquad (46.9)$$

Shifting four momenta

- At zero temperature, the integration is over 4-momenta K_μ
- The integration measure is invariant under $K_\mu \to K_\mu + \mathrm{const}$
- We can exploit this invariance by clearly shifting K
- In our case, using

$$K_\mu \to K_\mu - x_1 P_\mu - x_2 P'_\mu \tag{46.10}$$

we have

$$D \to K^2 + P^2 x_1(1-x_1) + P'^2 x_2(1-x_2) + m^2(x_1+x_2) - 2x_1x_2 P\cdot P'$$

Shifting four momenta

- Since the denominator depends now only on K^2, any terms linear in K in the numerator integrate to zero, and we find

$$N_\mu \to -\gamma_\sigma \not{K} \gamma_\mu \not{K} \gamma_\sigma + \gamma_\sigma \left(\not{a}_1 - m\right) \gamma_\mu \left(\not{a}_2 - m\right) \gamma_\sigma \,, \tag{46.11}$$

where $a_{1,\mu} = -ix_1 P_\mu + i(1-x_2)P'_\mu$, $a_{2,\mu} = i(1-x_1)P_\mu - ix_2 P'_\mu$

- Expanding the individual terms, we have

$$\begin{aligned} N_\mu &= -\gamma_\sigma \not{K} \gamma_\mu \not{K} \gamma_\sigma + \gamma_\sigma \not{a}_1 \gamma_\mu \not{a}_2 \gamma_\sigma + m^2 \gamma_\sigma \gamma_\mu \gamma_\sigma \\ &\quad -m \left(\gamma_\sigma \gamma_\mu \not{a}_2 \gamma_\sigma + \gamma_\sigma \not{a}_1 \gamma_\mu \gamma_\sigma\right) \,. \end{aligned} \tag{46.12}$$

- To reduce the number of γ matrices we use the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$

Gamma gynmastics

- For instance, we have

$$\gamma_\sigma \gamma_\mu \gamma_\sigma = -\gamma_\sigma^2 \gamma_\mu + 2\gamma_\sigma \delta_{\mu\sigma} = \gamma_\mu \left(2 - \mathrm{Tr}\, \mathbf{1}_4\right) = -2\gamma_\mu \qquad (46.13)$$

- Next, we consider

$$\begin{aligned} \gamma_\sigma \not{K} \gamma_\mu \not{K} \gamma_\sigma &= \gamma_\sigma \not{K} \gamma_\mu \left(-\gamma_\sigma \not{K} + 2K_\sigma\right) , \\ &= \gamma_\sigma \not{K} \gamma_\sigma \gamma_\mu \not{K} - 2\gamma_\mu \not{K}^2 + 2\gamma_\sigma \not{K} \gamma_\mu K_\sigma , \\ &= -\gamma_\sigma^2 \not{K} \gamma_\mu \not{K} + 2\not{K} \gamma_\mu \not{K} - 2\gamma_\mu K^2 + 2K^2 \gamma_\mu , \\ &= -2\not{K} \gamma_\mu \not{K} = 2K^2 \gamma_\mu - 4\not{K} K_\mu . \end{aligned} \qquad (46.14)$$

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- Next, we consider

$$\begin{aligned}\gamma_\sigma \not{a}_1 \gamma_\mu \not{a}_2 \gamma_\sigma &= -\gamma_\sigma \not{a}_1 \gamma_\mu \gamma_\sigma \not{a}_2 + 2\not{a}_2 \not{a}_1 \gamma_\mu \,, \\ &= \not{a}_1 \gamma_\sigma \gamma_\mu \gamma_\sigma \not{a}_2 - 2\gamma_\mu \not{a}_1 \not{a}_2 + 2\not{a}_2 \not{a}_1 \gamma_\mu \,, \\ &= -2\not{a}_1 \gamma_\mu \not{a}_2 - 2\gamma_\mu \not{a}_1 \not{a}_2 + 2\not{a}_2 \not{a}_1 \gamma_\mu \,, \\ &= -4a_{1,\mu} \not{a}_2 + 2\not{a}_2 \not{a}_1 \gamma_\mu = -2\not{a}_2 \gamma_\mu \not{a}_1 \,. \qquad (46.15)\end{aligned}$$

- Finally

$$\begin{aligned}\gamma_\sigma \gamma_\mu \not{a}_2 \gamma_\sigma + \gamma_\sigma \not{a}_1 \gamma_\mu \gamma_\sigma &= -\gamma_\sigma \gamma_\mu \gamma_\sigma \not{a}_2 + 2\not{a}_2 \gamma_\mu - \not{a}_1 \gamma_\sigma \gamma_\mu \gamma_\sigma + 2\gamma_\mu \not{a}_1 \,, \\ &= 2\gamma_\mu \not{a}_2 + 2\not{a}_2 \gamma_\mu + 2\not{a}_1 \gamma_\mu + 2\gamma_\mu \not{a}_1 \,, \\ &= 4(a_{1,\mu} + a_{2,\mu}) \qquad (46.16)\end{aligned}$$

Assembly

- Assembling the individual parts, we have

$$\Gamma_\mu^{(3),\text{conn.,amp.}}(P,P') = -2ie^3\int_0^1 dx_1 dx_2 \int_K \frac{N_\mu}{D^3}\,, \qquad (46.17)$$

where

$$D = K^2 + P^2 x_1(1-x_1) + P'^2 x_2(1-x_2) + m^2(x_1+x_2) - 2x_1x_2 P\cdot P'\,,$$
$$N_\mu = 2\not{K}\gamma_\mu\not{K} - 2\not{a}_2\gamma_\mu\not{a}_1 - 2m^2\gamma_\mu - 4m\left(a_{1,\mu}+a_{2,\mu}\right)\,. \qquad (46.18)$$

and

$$a_{1,\mu} = -ix_1P_\mu + i(1-x_2)P'_\mu\,,\quad a_{2,\mu} = i(1-x_1)P_\mu - ix_2P'_\mu\,. \quad (46.19)$$