g-2, part V

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Review

- In lecture 42, we discussed that g-2 is related to the connected and amputated photon-electron vertex $\Gamma_\mu^{\rm conn.,amp.}$
- In lecture 46, we derived the result

$$\Gamma_\mu^{(3),{\rm conn.,amp.}}(P,P') = -2ie^3 \int_0^1 dx_1 dx_2 \int_K \frac{N_\mu}{D^3}\,, \qquad (48.1)$$

- The explicit expressions for N_μ, D can be found in eqns. (46.18) in lecture 46
- The amputated vertex is sandwiched between two free fermion propagators,

$$S_{P'}\Gamma_\mu^{(3),{\rm conn.,amp.}}(P,P')S_P \qquad (48.2)$$

- In lecture 47, we discussed that the contribution $\not{K}\gamma_\mu\not{K}$ in N_μ does not contribute to $g-2$
- In this lecture, we will finish our calculation of g-2

K-integration

- The parts of N_μ contributing to g-2 are given by

$$N_\mu = -2\not{a}_2\gamma_\mu\not{a}_1 - 2m^2\gamma_\mu - 4m\left(a_{1,\mu} + a_{2,\mu}\right) \tag{48.3}$$

- The form of the denominator is given by $D = K^2 + A$, where on-shell

$$A = m^2(x_1 + x_2)^2 + x_1 x_2 Q^2\,. \tag{48.4}$$

  and $Q_\mu = P'_\mu - P_\mu$.
- The K-integration is straightforward using the method from lecture 10:

$$\int_K \frac{N_\mu}{D^3} = \frac{1}{2(4\pi)^2}\frac{N_\mu}{A}\,. \tag{48.5}$$

Relevant vertex

- One finds for the part of the vertex relevant for g-2:

$$\Gamma_\mu = \frac{ie^3}{16\pi^2}\int_0^1 dx_1 dx_2 \frac{2\not{a}_2\gamma_\mu\not{a}_1 + 2m^2\gamma_\mu + 4m\left(a_{1,\mu}+a_{2,\mu}\right)}{m^2(x_1+x_2)^2 + x_1x_2Q^2} \quad (48.6)$$

- Now we use the fact that Γ_μ is sandwiched between propagators (48.2),
- For instance,

$$\begin{aligned} \not{a}_1 S(P) &= \left(-ix_1\not{P} + i(1-x_2)\not{P}'\right) S(P)\,, \\ &\to \left(mx_1 + i(1-x_2)\not{P}'\right) S(P)\,, \end{aligned} \quad (48.7)$$

because

$$i\not{P}S(P) = \frac{i\not{P}}{i\not{P}+m} = 1 - mS(P)\,. \quad (48.8)$$

Gamma gymnastics

- Next, pull through γ_μ:

$$\gamma_\mu \not{a}_1 = \left(m x_1 \gamma_\mu + 2i(1-x_2)P'_\mu - (1-x_2) i \not{P}' \gamma_\mu \right) \quad (48.9)$$

- We can do a similar trick with $\not{a}_2$ acting to the left so that

$$\begin{aligned}
\not{a}_2 \gamma_\mu \not{a}_1 &\rightarrow \left(i(1-x_1)\not{P} + m x_2 \right) \gamma_\mu \not{a}_1 \\
&= (1-x_1)\left(m x_1 i \not{P} \gamma_\mu - 2im(1-x_2)P'_\mu - (1-x_2) i \not{P} i \not{P}' \gamma_\mu \right) \\
&\quad + m x_2 \left(m x_1 \gamma_\mu + 2i(1-x_2)P'_\mu + m(1-x_2)\gamma_\mu \right) , \\
&= (1-x_1)\left(-2im(1-x_2)P'_\mu - m(1-x_1-x_2) i \not{P} \gamma_\mu \right) \\
&\quad + m x_2 \left(2i(1-x_2)P'_\mu + m(1+x_1-x_2)\gamma_\mu \right) \\
&\quad + 2(1-x_1)(1-x_2) P \cdot P' \gamma_\mu \qquad (48.10)
\end{aligned}$$

Gamma gymnastics cnt'ed

- In the next step we have

$$
\begin{aligned}
\not{a}_2\gamma_\mu\not{a}_1 \quad \rightarrow \quad & (1-x_1)\left(-2im(1-x_2)P'_\mu - m^2(1-x_1-x_2)\gamma_\mu\right)\\
& -2m(1-x_1)(1-x_1-x_2)iP_\mu\\
& +mx_2\left(2i(1-x_2)P'_\mu + m(1+x_1-x_2)\gamma_\mu\right)\\
& +2(1-x_1)(1-x_2)P\cdot P'\gamma_\mu\\
= \quad & -m^2\gamma_\mu\left(1+x_1^2+x_2^2-2x_1-2x_2\right)\\
& +2(1-x_1)(1-x_2)P\cdot P'\gamma_\mu\\
& -2imP'_\mu(1-x_2)(1-x_1-x_2)\\
& -2imP_\mu(1-x_1)(1-x_1-x_2) \qquad (48.11)
\end{aligned}
$$

Vertex Numerator

- Plugging this result into the numerator (48.3), we have

$$
\begin{aligned}
N_\mu \;=\;& 2m^2\gamma_\mu\left(x_1^2+x_2^2-2x_1-2x_2\right)\\
&-4(1-x_1)(1-x_2)P\cdot P'\gamma_\mu\\
&+4imP'_\mu(1-x_2)(1-x_1-x_2)\\
&+4imP_\mu(1-x_1)(1-x_1-x_2)\\
&-4imP_\mu(1-2x_1)-4imP'_\mu(1-2x_2)\,. \qquad (48.12)
\end{aligned}
$$

- Now write

$$2P\cdot P' = \left[-Q^2+P^2+P'^2\right] = \left[-Q^2-2m^2\right] \qquad (48.13)$$

and

$$P'_\mu = \frac{1}{2}\left(P'_\mu+P_\mu\right)+\frac{1}{2}\left(P'_\mu-P_\mu\right) \qquad (48.14)$$

Vertex Numerator

- This gives

$$
\begin{aligned}
N_\mu &= 2m^2\gamma_\mu\left(x_1^2+x_2^2-2x_1-2x_2\right)\\
&\quad 2(1-x_1)(1-x_2)\left(Q^2+2m^2\right)\gamma_\mu\\
&\quad +2im\left(P'_\mu+P_\mu\right)\left(x_1^2+x_2^2+2x_1x_2-x_1-x_2\right)\\
&\quad -2im\left(P'-P_\mu\right)\left(x_1^2-x_2^2+x_1-x_2\right)
\end{aligned}
$$

- This expression can be simplified using $x_3=1-x_1-x_2$ by noting that

$$
x_1^2+x_2^2+2x_1x_2-x_1-x_2=x_3^2-x_3 \tag{48.15}
$$

such that

$$
\begin{aligned}
N_\mu &= 2m^2\gamma_\mu\left(x_3^2+2x_3-1\right)+2(1-x_1)(1-x_2)Q^2\gamma_\mu\\
&\quad +2im\left(P'_\mu+P_\mu\right)\left(x_3^2-x_3\right)\\
&\quad -2im\left(P'-P_\mu\right)\left(x_1^2-x_2^2+x_1-x_2\right)
\end{aligned} \tag{48.16}
$$

Symmetries of the integrand

- From (48.6), we now have

$$\Gamma_\mu = -\frac{ie^3}{16\pi^2}\int_0^1 dx_1 dx_2 \frac{N_\mu}{m^2(x_1+x_2)^2 + x_1 x_2 Q^2}\,. \qquad (48.17)$$

- Since the denominator and measure are invariant under $x_1 \leftrightarrow x_2$, the last term in (48.16) cancels out
- For g-2 we are looking for a contribution that is linear in Q, so the term proportional to Q^2 also drops out
- We are left with

$$\Gamma_\mu = \frac{ie^3}{16\pi^2}\int_0^1 dx_1 dx_2 \frac{2m^2\gamma_\mu\left(1-2x_3-x_3^2\right) + 2im\left(P'_\mu + P_\mu\right)\left(x_3 - x_3^2\right)}{m^2(x_1+x_2)^2 + x_1 x_2 Q^2} \qquad (48.18)$$

Gordon identity

- We now use the Gordon identity in momentum space (42.12) to replace

$$i(P'_\mu + P_\mu) \to -2m\gamma_\mu + \sigma_{\mu\nu} Q_\nu \tag{48.19}$$

- Only the term proportional to $\sigma_{\mu\nu}$ will contribute to g-2
- Using the Gordon identity in (48.18), we have for the part of the vertex relevant for g-2:

$$\Gamma_\mu = \frac{ie^3}{16\pi^2} \int_0^1 dx_1 dx_2 \frac{2m\sigma_{\mu\nu} Q_\nu \left(x_3 - x_3^2\right)}{m^2(x_1+x_2)^2 + x_1 x_2 Q^2} \,. \tag{48.20}$$

- Comparing this with the definition of the vertex (42.10)

$$\Gamma_\mu = \frac{ie}{2m} \sigma_{\mu\nu} Q_\nu F_2(Q^2) \tag{48.21}$$

we find an expression for the form factor F_2

Form Factor F_2

- We find for the form factor F_2:

$$F_2(Q^2) = \frac{e^2}{4\pi^2}\int_0^1 dx_1 dx_2 \frac{\left(x_3 - x_3^2\right)}{(x_1+x_2)^2 + x_1 x_2 Q^2/m^2}\,. \tag{48.22}$$

- Rewrite using $\int_0^1 dx_3 \delta(1-x_1-x_2-x_3)$ and do x_2-integration

$$F_2(Q^2) = \frac{e^2}{4\pi^2}\int_0^1 dx_1 dx_3 \frac{\left(x_3 - x_3^2\right)}{(1-x_3)^2 + x_1(1-x_1-x_3)Q^2/m^2}\,. \tag{48.23}$$

- Write $x_1 = (1-x_3)y$ to find

$$F_2(Q^2) = \frac{e^2}{4\pi^2}\int_0^1 dy dx_3 \frac{x_3\left(1-x_3\right)^2}{(1-x_3)^2 + (1-x_3)^2 y(1-y)Q^2/m^2}\,. \tag{48.24}$$

Form Factor F_2

- Perform the integral over x_3:

$$F_2(Q^2) = \frac{e^2}{8\pi^2} \int_0^1 dy \frac{1}{1 + y(1-y)Q^2/m^2} \,. \tag{48.25}$$

- Recall from lecture 42 that we only need $F_2(0)$ for g-2:
- We get

$$g - 2 = 2F_2(0) = \frac{e^2}{4\pi^2} = \frac{\alpha}{\pi} \,. \tag{48.26}$$

- Here $\alpha = \frac{1}{137.036}$ is the fine-structure constant

g-2

- Expressing g-2 as

$$g - 2 = 2(1 + a)\,, \tag{48.27}$$

our calculation gives

$$a = \frac{\alpha}{2\pi} \simeq 0.00116141\,. \tag{48.28}$$

- A recent experimental value [Hanneke et al. 2008] gives

$$a = 0.00115965218073(28)\,. \tag{48.29}$$

- Our calculation agrees with experiment to 3 significant digits!