ALLGEMEINE RELATIVITÄTSTHEORIE

$$ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$$

$g_{\mu\nu}(x)$ nichtsingulärer ($\det g_{\mu\nu} \neq 0$) symmetrischer Tensor
$n=4$ 10 unabhängige Komponenten

allgemeine Koordinatentransformationen $x^\mu \to \bar{x}^\mu(x^\nu)$

$$ds^2 = g_{\alpha\beta}(x)\, dx^\alpha dx^\beta = \bar{g}_{\mu\nu}(\bar{x})\, d\bar{x}^\mu d\bar{x}^\nu$$

$$\bar{g}_{\mu\nu}(\bar{x}) = g_{\alpha\beta}(x) \frac{dx^\alpha}{d\bar{x}^\mu} \frac{dx^\beta}{d\bar{x}^\nu}$$

Bewegung im Gravitationsfeld

$$0 = \delta \int ds = \delta \int \sqrt{g_{\mu\nu}(x)\, \dot{x}^\mu \dot{x}^\nu}\, d\lambda \equiv \delta \int \mathcal{L}\, d\lambda = 0 \qquad \dot{x}^\mu = \frac{dx^\mu}{d\lambda}$$

Eulergleichungen des Variationsproblems $\delta \int \mathcal{L}\, d\lambda = 0$

$$\frac{\partial \mathcal{L}}{\partial x^\mu} - \frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = 0$$

Geodätengleichung

$$g_{\mu\nu}\, \ddot{x}^\nu + \Gamma_{\mu\nu\lambda}\, \dot{x}^\nu \dot{x}^\lambda = 0$$

Christoffel-Symbole $\Gamma_{\mu\nu\lambda} = \frac{1}{2}\left(g_{\mu\nu,\lambda} + g_{\mu\lambda,\nu} - g_{\nu\lambda,\mu}\right)$

$\hookrightarrow$ $\Gamma_{\mu\nu\lambda} = \Gamma_{\mu\lambda\nu}$

$\hookrightarrow$ $\Gamma_{\mu\nu\lambda} + \Gamma_{\lambda\nu\mu} = g_{\mu\lambda,\nu}$

$$\Gamma^\mu{}_{\nu\lambda} = g^{\mu\sigma}\, \Gamma_{\sigma\nu\lambda} \qquad g^{\mu\sigma} g_{\sigma\lambda} = \delta^\mu_\lambda \qquad A_\mu = g_{\mu\nu} A^\nu \qquad A^\mu = g^{\mu\nu} A_\nu$$

Ableitung der Determinante $g = \det(g_{\mu\nu})$

$$\frac{dg}{dx^\mu} = g \cdot g^{\nu\lambda} \frac{dg_{\nu\lambda}}{dx^\mu} \qquad \frac{1}{g} \frac{dg}{dx^\mu} = 2\Gamma^\nu{}_{\nu\mu}$$

$$dg = dg_{\alpha\beta}\, M^{\alpha\beta}$$

Inverse $g^{\alpha\beta} = (g_{\alpha\beta})^{-1} = \frac{1}{g} M^{\alpha\beta}$

$\hookrightarrow$ $dg = dg_{\alpha\beta}\left(g \cdot g^{\alpha\beta}\right) \Rightarrow \frac{dg}{dx^\mu} = g \cdot g^{\alpha\beta} \frac{dg_{\alpha\beta}}{dx^\mu}$

Invariantes Volumselement

$$\bar{g}_{\mu\nu}(\bar{x}) = \frac{\partial x^\alpha}{\partial \bar{x}^\mu} \frac{\partial x^\beta}{\partial \bar{x}^\nu} g_{\alpha\beta}(x) \Rightarrow \det \bar{g} = J^2 \det g$$

$$J = \det\left(\frac{\partial x}{\partial \bar{x}}\right) \Rightarrow \bar{g} = J^2 \cdot g \qquad \sqrt{-\bar{g}} = J \cdot \sqrt{-g}$$

Koordinatentransformation $x \to \bar{x}$ $\Rightarrow$ $d^4x = J \cdot d^4\bar{x}$

$$\Rightarrow d^4\bar{x} \cdot \sqrt{-\bar{g}} = J^{-1} d^4x \cdot J \sqrt{-g} = d^4x \cdot \sqrt{-g}$$

Tensoren im Riemann'schen Raum

Die Form der Gleichungen soll bei Koordinatentransformation gleich bleiben

$\Downarrow$

Formulierung mittels Tensoren (definiert durch ihr Verhalten bei Koordinatentransformationen $x^\mu \to \bar{x}^\mu(x)$

Skalar $\quad \bar{S}(\bar{x}) = S(x)$

Vektor $\quad \bar{A}^\mu(\bar{x}) = \dfrac{\partial \bar{x}^\mu}{\partial x^\alpha} A^\alpha(x) \qquad \bar{A}_\mu(\bar{x}) = \dfrac{\partial x^\alpha}{\partial \bar{x}^\mu} A_\alpha(x)$

$\downarrow$ kontravariant $\qquad \downarrow$ kovariant

generell:

$$\underline{\bar{T}^{\mu\nu\ldots}{}_{\alpha\beta\ldots}(\bar{x})} = \underline{\left(\frac{\partial \bar{x}^\mu}{\partial x^\lambda}\right)\left(\frac{\partial \bar{x}^\nu}{\partial x^\sigma}\right)\cdots\left(\frac{\partial x^\gamma}{\partial \bar{x}^\alpha}\right)\left(\frac{\partial x^\delta}{\partial \bar{x}^\beta}\right)\cdots T^{\lambda\sigma\ldots}{}_{\gamma\delta\ldots}(x)}$$

Typ (m/n) $\qquad \mu\nu\ldots$ m Indizes
$\qquad \alpha\beta\ldots$ n " – "

kovariante Ableitung

Die Ableitung eines Vektors $\dfrac{\partial A^\mu(x)}{\partial x^\nu} = A^\mu{}_{,\nu}$ transformiert sich nicht wie ein Tensor:

$$\bar{A}^\beta{}_{,\alpha} = \frac{\partial \bar{A}^\beta(\bar{x})}{\partial \bar{x}^\alpha} = \frac{\partial x^\mu}{\partial \bar{x}^\alpha}\cdot\frac{\partial}{\partial x^\mu}\left(\frac{\partial \bar{x}^\beta}{\partial x^\lambda} A^\lambda(x)\right)$$

$$= \underbrace{\left(\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\right)\left(\frac{\partial^2 \bar{x}^\beta}{\partial x^\mu \partial x^\lambda}\right) A^\lambda(x)}_{\text{„störender" Term}} + \underbrace{\left(\frac{\partial x^\mu}{\partial \bar{x}^\alpha}\right)\left(\frac{\partial \bar{x}^\beta}{\partial x^\lambda}\right)}_{\text{„richtiges" Transformationsverhalten}} A^\lambda{}_{,\mu}(x)$$

kovariante Ableitung $\hat{=}\ ;$

Skalar $\quad \underline{\phi_{;\mu} = \phi_{,\mu}}$

Vektor $\quad \underline{\underline{A^\mu{}_{;\nu}}} = \underline{A^\mu{}_{,\nu} + \Gamma^\mu{}_{\nu\lambda} A^\lambda} \qquad \underline{A_{\mu;\nu}} = \underline{A_{\mu,\nu} - \Gamma^\lambda{}_{\mu\nu} A_\lambda}$

Tensor Typ (2/0) $\quad \underline{T^{\mu\nu}{}_{;\lambda}} = \underline{T^{\mu\nu}{}_{,\lambda} + \Gamma^\mu{}_{\lambda\sigma} T^{\sigma\nu} + \Gamma^\nu{}_{\lambda\sigma} T^{\mu\sigma}}$

kovariante Ableitung der Metrik verschwindet $\quad \underline{\underline{g_{\mu\nu;\lambda} = 0}}$

$$g_{\mu\nu;\lambda} = g_{\mu\nu,\lambda} - \Gamma^\rho{}_{\mu\lambda} g_{\rho\nu} - \Gamma^\rho{}_{\nu\lambda} g_{\mu\rho} = g_{\mu\nu,\lambda} - (\Gamma_{\nu\mu\lambda} + \Gamma_{\mu\nu\lambda})$$

$$= g_{\mu\nu,\lambda} - \underbrace{(\Gamma_{\nu\lambda\mu} + \Gamma_{\mu\lambda\nu})}_{g_{\mu\nu,\lambda}} = \underline{\underline{0}}$$

$\underline{g_{\mu\nu;\lambda} = 0} \qquad (g^{\mu\rho} g_{\rho\nu})_{;\lambda} = (\delta^\mu_\nu)_{;\lambda} = 0 \implies \underline{g^{\mu\nu}{}_{;\lambda} = 0}$

$\underline{A^{\mu;\nu} = g^{\nu\lambda} A^\mu{}_{;\lambda}}$

$\boxed{\text{Riemann'scher Krümmungstensor } R_{\mu\nu\lambda\sigma}}$

Differentiationsreihenfolge im Falle von kovarianten (im Gegensatz zu partiellen) Ableitungen vertauschen im allgemeinen nicht!

↳ für einen <u>Skalar</u> lassen sich die beiden Ableitungen vertauschen

$$\phi_{;\mu;\nu} = (\phi_{,\mu})_{;\nu} = \phi_{,\mu\nu} - \Gamma^{\lambda}{}_{\mu\nu}\,\phi_{,\lambda} = \phi_{;\nu;\mu}$$

↳ für einen <u>Vektor</u> ist das im allgemeinen nicht der Fall.

$$\underline{A_{\mu;\nu;\lambda} - A_{\mu;\lambda;\nu}} = \dots\dots = \underline{A_\sigma R^{\sigma}{}_{\mu\nu\lambda}} \qquad \text{Ricci-Identität}$$

mit

$$\underline{R^{\mu}{}_{\nu\sigma\lambda}} = \Gamma^{\mu}{}_{\nu\lambda,\sigma} - \Gamma^{\mu}{}_{\nu\sigma,\lambda} + \Gamma^{\mu}{}_{\alpha\sigma}\Gamma^{\alpha}{}_{\lambda\nu} - \Gamma^{\mu}{}_{\alpha\lambda}\Gamma^{\alpha}{}_{\sigma\nu}$$

$$\Downarrow$$

$$\boxed{R_{\lambda\mu\nu\kappa}} = \underline{\frac{1}{2}\left(g_{\lambda\kappa,\mu\nu} + g_{\mu\nu,\lambda\kappa} - g_{\mu\kappa,\nu\lambda} - g_{\nu\lambda,\mu\kappa}\right) + \Gamma^{\alpha}{}_{\mu\nu}\Gamma_{\alpha\lambda\kappa} - \Gamma^{\alpha}{}_{\mu\kappa}\Gamma_{\alpha\lambda\nu}}$$

<u>Symmetrieeigenschaften</u>

(a) $R_{\lambda\mu\nu\kappa} = -R_{\mu\lambda\nu\kappa}$

(b) $R_{\lambda\mu\nu\kappa} = -R_{\lambda\mu\kappa\nu}$

(c) $R_{\lambda\mu\kappa\nu} = R_{\kappa\nu\lambda\mu}$

(d) $R_{\lambda\mu\kappa\nu} + R_{\lambda\nu\mu\kappa} + R_{\lambda\kappa\nu\mu} = 0$

(c) folgt aus (a), (b) & (d)

$\Rightarrow$ in n Dimensionen $\binom{n}{2}\cdot\binom{n}{2} - n\binom{n}{3} = \dfrac{n^2(n^2-1)}{12}$ unabhängige Komponenten ($n=4 \;\hat{=}\; 20$)

<u>Ricci-tensor</u> $\quad \underline{R_{\mu\nu} \equiv R^{\lambda}{}_{\mu\lambda\nu} = R_{\nu\mu}}$

<u>Krümmungsskalar</u> $\quad \underline{R \equiv g^{\mu\nu} R_{\mu\nu}}$

<u>Bianchi-Identitäten</u>

$$\underline{R^{\lambda}{}_{\mu\nu\sigma;\kappa} + R^{\lambda}{}_{\mu\kappa\nu;\sigma} + R^{\lambda}{}_{\mu\sigma\kappa;\nu} = 0}$$

(ergibt sich aus kovarianter Differentiation von $R^{\lambda}{}_{\mu\nu\sigma}$)

Kontraktion λ & ν $\Rightarrow$

$$\underbrace{R^{\lambda}{}_{\mu\lambda\sigma;\kappa}}_{R_{\mu\sigma;\kappa}} + \underbrace{R^{\lambda}{}_{\mu\kappa\lambda;\sigma}}_{-R_{\mu\kappa;\sigma}} + R^{\lambda}{}_{\mu\sigma\kappa;\lambda} = 0$$

$$R_{\mu\sigma;\kappa} - R_{\mu\kappa;\sigma} + R^{\lambda}{}_{\mu\sigma\kappa;\lambda} = 0 \qquad \Big|\cdot g^{\mu\kappa}$$

$$R^{\kappa}{}_{\sigma;\kappa} - R_{;\sigma} + \underbrace{R^{\lambda\kappa}{}_{\sigma\kappa;\lambda}}_{\hat{=}\; R^{\kappa\lambda}{}_{\kappa\sigma;\lambda} \,=\, R^{\lambda}{}_{\sigma;\lambda}} = 0$$

$$\Rightarrow \quad \underline{2R^{\kappa}{}_{\sigma;\kappa} - R_{;\sigma} = 0} \qquad \underline{R^{\kappa}{}_{\sigma;\kappa} = \frac{1}{2} R_{;\sigma}}$$

<u>Einsteintensor</u> $\quad \underline{G^{\mu}{}_{\nu} = R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}_{\nu} R}$

$$\underline{G^{\mu}{}_{\nu;\mu} = 0}$$

Einstein'sche Feldgleichungen

Herleitung aus Variationsprinzip

$$\underline{\underline{S_g}} = \frac{-1}{16\pi G} \int \underbrace{d^4x \sqrt{-g}}_{\text{invariantes Volumselement}} (R + 2\Lambda)$$

R: skalar

(enthält zunächst 2.te Ableitungen der Metrik, eliminieren durch partielle $\int$)

für die materie

$$\underline{\underline{S_{mat}}} = \int d^4x \sqrt{-g}\, \mathcal{L}_{mat}(\text{Feldern}, g_{\mu\nu})$$

$$\underline{S_{tot} = S_g + S_{mat}}$$

$$\underline{\delta S_{tot} = 0} \qquad \text{Euler-Lagrange}: \quad \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} - \frac{d}{dx^\alpha} \frac{\partial \mathcal{L}}{\partial g_{\mu\nu,\alpha}} = 0$$

$$\Downarrow$$

$$\underline{G_{\mu\nu}} = \underline{\underline{R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G\, T_{\mu\nu} + \Lambda g_{\mu\nu}}}$$

mit

$$\underline{\underline{T_{\mu\nu}}} \equiv \underline{\underline{\frac{2}{\sqrt{-g}} \left[\frac{\partial(\sqrt{-g}\,\mathcal{L}_{mat})}{\partial g^{\mu\nu}} - \frac{d}{dx^\alpha} \frac{\partial(\sqrt{-g}\,\mathcal{L}_{mat})}{\partial g^{\mu\nu}{}_{,\alpha}} \right]}}$$

Energie – Impulstensor

$$\underline{\delta S_{mat}} = \int d^4x \sqrt{-g}\, \frac{1}{2} T_{\mu\nu}\, \delta g^{\mu\nu}$$