SYMMETRIC SPACES

How can we use some supposed symmetry of a metric space to gain information about the metric?

↳ Forminvariance of the metric : $\underline{g'_{\mu\nu}(x') = g_{\mu\nu}(x') \quad \forall x'}$ (1)

transformed metric : $g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) \stackrel{!}{=} g_{\mu\nu}(x')$

$$\Downarrow$$

$$\underline{g_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu} \frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\sigma}(x')} \qquad (2)$$

⟶ Any transformation $x^\mu \to x'^\mu$, that satisfies (2) is called an <u>isometry</u>

<u>infinitesimal coordinate transformations</u> $x'^\mu = x^\mu + \varepsilon \xi^\mu(x) \quad |\varepsilon| \ll 1$ (3)

$$\underset{\sim\sim}{O(\varepsilon)} \quad g_{\mu\nu}(x) = \left(\delta^\rho_\mu + \varepsilon \xi^\rho{}_{,\mu}\right)\left(\delta^\sigma_\nu + \varepsilon \xi^\sigma{}_{,\nu}\right)\left(g_{\rho\sigma}(x) + \varepsilon \xi^\lambda g_{\rho\sigma,\lambda} + \dots\right)$$

$$= g_{\mu\nu}(x) + \varepsilon g_{\rho\nu} \xi^\rho{}_{,\mu}(x) + \varepsilon g_{\mu\sigma} \xi^\sigma{}_{,\nu} + \varepsilon \xi^\lambda g_{\mu\nu,\lambda}$$

$$\Downarrow$$

$$\underline{0 = \xi^\rho{}_{,\mu} g_{\rho\nu} + \xi^\rho{}_{,\nu} g_{\mu\rho} + \xi^\lambda g_{\mu\nu,\lambda}} \qquad (4)$$

this can be rewritten in terms of $\underline{\xi_\mu = g_{\mu\nu}\xi^\nu}$

$$0 = \frac{\partial}{\partial x^\mu}\underbrace{\left(\xi^\rho g_{\rho\nu}\right)}_{\xi_\nu} + \frac{\partial}{\partial x^\nu}\underbrace{\left(g_{\mu\rho}\xi^\rho\right)}_{\xi_\mu} \underbrace{- \xi^\rho g_{\rho\nu,\mu} - \xi^\rho g_{\mu\rho,\nu} + \xi^\lambda g_{\mu\nu,\lambda}}_{+\xi^\lambda \underbrace{\left(g_{\mu\nu,\lambda} - g_{\lambda\nu,\mu} - g_{\mu\lambda,\nu}\right)}_{(-2)\Gamma_{\lambda\mu\nu} = (-2)\Gamma_{\lambda\nu\mu}}}$$

$$\Rightarrow \quad 0 = \xi_{\mu,\nu} - \xi^\lambda \Gamma_{\lambda\mu\nu} + \xi_{\nu,\mu} - \xi^\lambda \Gamma_{\lambda\nu\mu} = \xi_{\mu;\nu} + \xi_{\nu;\mu}$$

$$\underline{\underline{0 = \xi_{\mu;\nu} + \xi_{\nu;\mu}}} \qquad (5)$$

Any vector, that satisfies (4) or (5) is said to form a <u>Killingvector</u> of the metric $g_{\mu\nu}(x)$

↳ Killing condition (5) much more <u>restrictive</u> than it looks
allows to determine the whole function $\xi_\mu(x)$ from any given values ξ_μ and $\xi_{\mu;\nu}$ at some point X

commutator of 2 covariant derivatives : $\underline{\xi_{\sigma;\rho;\mu} - \xi_{\sigma;\mu;\rho} = -R^{\lambda}{}_{\sigma\rho\mu}\xi_\lambda}$ (6)

cyclic sum rule for curvature tensor : $\underline{R^{\lambda}{}_{\sigma\rho\mu} + R^{\lambda}{}_{\mu\sigma\rho} + R^{\lambda}{}_{\rho\mu\sigma} = 0}$ (7) $\Big| \cdot \xi_\lambda$

$$\Rightarrow \xi_{\sigma;\rho;\mu} - \xi_{\sigma;\mu;\rho} + \underbrace{\xi_{\mu;\sigma;\rho}}_{(-\xi_{\sigma;\mu;\rho})} - \xi_{\mu;\rho;\sigma} + \underbrace{\xi_{\rho;\mu;\sigma}}_{(-\xi_{\mu;\rho;\sigma})} - \underbrace{\xi_{\rho;\sigma;\mu}}_{+\xi_{\sigma;\rho;\mu}} = 0$$

$\Downarrow$ ($\xi_{\rho;\sigma} + \xi_{\sigma;\rho} = 0$!)

$$2\,[\,\xi_{\sigma;\rho;\mu} - \xi_{\mu;\rho;\sigma} - \xi_{\sigma;\mu;\rho}\,] = 0$$

$$\xi_{\sigma;\rho;\mu} - \xi_{\sigma;\mu;\rho} = -R^{\lambda}{}_{\sigma\rho\mu}\xi_\lambda$$

$\Downarrow$

$$\underline{\xi_{\mu;\rho;\sigma} = -R^{\lambda}{}_{\sigma\rho\mu}\xi_\lambda} \qquad (8)$$

If ξ_μ and $\xi_{\mu;\nu}$ are known at a typical point X, all higher derivatives of ξ_μ at X can be determined and hence the entire function $\xi_\mu(x)$ in a neighbourhood of X by Taylorexpansion:

$$\underline{\xi_\lambda(x) = A_\lambda{}^{\rho}(x, X, g)\,\xi_\rho(X) + B_\lambda{}^{\rho\sigma}(x, X, g)\,\xi_{\rho;\sigma}(X)} \qquad (9)$$

$\Updownarrow$

$$B_\lambda{}^{\rho\sigma} = -B_\lambda{}^{\sigma\rho}$$

→ In a spacetime of N dimensions there are up to N quantities $\xi_\lambda(X)$ and up to $\frac{1}{2}N(N-1)$ quantities $\xi_{\mu;\nu}(X) = -\xi_{\nu;\mu}(X)$

$\Downarrow$

<u>maximally</u> $N + \frac{1}{2}N(N-1) = \underline{\frac{1}{2}N(N+1)}$ <u>Killingvectors</u> in <u>N dimensions</u>

<u>homogeneity</u>

A spacetime is said to be homogeneous, if there are infinitesimal isometries, that carry any given point X into any other point in its neighbourhood.

⇒ the metric must admit Killing vectors, that at any given point take on all possible values for $\xi_\mu(X)$ $\hat{=}$ N Killing-vectors in N dimensions

$\xi_\mu^{(n)}(X)$ $n = 1, 2, \ldots, N$

<u>isotropy</u>

A spacetime is said to be isotropic at a given point X, if there are Killingvectors ξ_μ for which $\xi_\mu(X) = 0$ and $\xi_{\mu;\nu}(X)$ take on all possible values $\hat{=}$ $\frac{1}{2}N(N-1)$

→ not all metrics admit maximum number of Killing vectors

Whether $\xi_{\mu;\nu;\rho} = -R^{\sigma}{}_{\rho\nu\mu}\,\xi_\sigma$ is soluble for a given set of initial data $\xi_\lambda(X)$, $\xi_{\lambda;\sigma}(X)$ depends on the integrability of this equation

→ <u>commutator of covariant derivatives of tensors:</u>

$$T_{\alpha\mu;\nu;\kappa} - T_{\alpha\mu;\kappa;\nu} = -T_{\sigma\mu}\,R^{\sigma}{}_{\alpha\nu\kappa} - T_{\alpha\sigma}\,R^{\sigma}{}_{\mu\nu\kappa}$$

∘) $\xi_{\rho;\mu;\nu;\kappa} - \xi_{\rho;\mu;\kappa;\nu} = -\xi_{\lambda;\mu}\,R^{\lambda}{}_{\rho\sigma\nu} - \xi_{\rho;\lambda}\,R^{\lambda}{}_{\mu\sigma\nu}$

∘) $\xi_{\rho;\mu;\sigma} = -R^{\lambda}{}_{\sigma\mu\rho}\,\xi_\lambda$

$$\underline{\underline{\left[-R^{\lambda}{}_{\rho\sigma\nu}\,\delta^{\kappa}_{\mu} + R^{\lambda}{}_{\mu\sigma\nu}\,\delta^{\kappa}_{\rho} + R^{\lambda}{}_{\nu\rho\mu}\,\delta^{\kappa}_{\sigma} - R^{\lambda}{}_{\sigma\rho\mu}\,\delta^{\kappa}_{\nu}\right]\xi_{\lambda;\kappa} =}}$$

$$\underline{\underline{= \left(R^{\lambda}{}_{\sigma\rho\mu;\nu} - R^{\lambda}{}_{\nu\rho\mu;\sigma}\right)\xi_\lambda}} \qquad (10)$$

maximally symmetric space : can find Killing vectors, for which $\xi_\mu(X) = 0$ & $\xi_{\mu;\lambda}$ arbitrary antisymm. matrix

⇓

coefficient in (10) of $\xi_{\lambda;\kappa}$ must have a vanishing antisymmetric part

$[\;]_{\lambda\kappa} - [\;]_{\kappa\lambda} = 0$

⇓

$$-R^{\lambda}{}_{\rho\nu\sigma}\,\delta^{\kappa}_{\mu} + R^{\lambda}{}_{\mu\sigma\nu}\,\delta^{\kappa}_{\rho} + R^{\lambda}{}_{\nu\rho\mu}\,\delta^{\kappa}_{\sigma} - R^{\lambda}{}_{\sigma\rho\mu}\,\delta^{\kappa}_{\nu} =$$

$$= -R^{\kappa}{}_{\rho\nu\sigma}\,\delta^{\lambda}_{\mu} + R^{\kappa}{}_{\mu\sigma\nu}\,\delta^{\lambda}_{\rho} + R^{\kappa}{}_{\nu\rho\mu}\,\delta^{\lambda}_{\sigma} - R^{\kappa}{}_{\sigma\rho\mu}\,\delta^{\lambda}_{\nu} \qquad \Big|\cdot\delta^{\mu}_{\kappa}$$

$$\underbrace{-NR^{\lambda}{}_{\rho\nu\sigma} - R^{\lambda}{}_{\rho\nu\sigma} + \underbrace{R^{\lambda}{}_{\nu\rho\sigma}}_{-R^{\lambda}{}_{\nu\sigma\rho}} - R^{\lambda}{}_{\sigma\rho\nu}}_{\theta} = -R^{\lambda}{}_{\rho\nu\sigma} + 0 + \underbrace{R^{\mu}{}_{\nu\rho\mu}\,\delta^{\lambda}_{\sigma} - R^{\mu}{}_{\sigma\rho\mu}\,\delta^{\lambda}_{\nu}}_{-R_{\nu\rho}\,\delta^{\lambda}_{\sigma} + R_{\sigma\rho}\,\delta^{\lambda}_{\nu}}$$

(with $R^{\mu}{}_{\mu\sigma\nu} = 0$ $\qquad R_{\sigma\rho} = R^{\mu}{}_{\sigma\mu\rho} = -R^{\mu}{}_{\sigma\rho\mu}$)

$(N-1)\,R^{\lambda}{}_{\rho\sigma\nu} = R_{\nu\rho}\,\delta^{\lambda}_{\sigma} - R_{\sigma\rho}\,\delta^{\lambda}_{\nu} \qquad \big|\cdot g_{\alpha\lambda}$

$$\underline{(N-1)\,R_{\alpha\rho\sigma\nu} = \underbrace{g_{\alpha\sigma}R_{\nu\rho} - g_{\alpha\nu}R_{\sigma\rho}}_{\text{must be antisymmetric in } (\alpha\rho)}} \qquad (13)$$

$(N-1)\,R_{\alpha\rho\sigma\nu} = -g_{\rho\sigma}R_{\nu\alpha} + g_{\rho\nu}R_{\sigma\alpha} \qquad \big|\cdot g^{\alpha\nu}$

$(N-1)(-R_{\rho\sigma}) = -g_{\rho\sigma}R^{\lambda}{}_{\lambda} + R_{\sigma\rho}$

$$\Rightarrow \quad \underline{\underline{R_{\sigma\rho} = \frac{1}{N}\,g_{\sigma\rho}\,R^{\lambda}{}_{\lambda} \qquad (14)}}$$

Inserting (14) into (13)

$$(N-1)\, R_{\lambda\rho\sigma\nu} = \frac{1}{N} R^{\lambda}{}_{\lambda} \left(g_{\lambda\sigma} g_{\nu\rho} - g_{\lambda\nu} g_{\sigma\rho} \right)$$

$$\underline{R_{\lambda\rho\sigma\nu}} = \underline{\frac{R^{\lambda}{}_{\lambda}}{N(N-1)} \left(g_{\lambda\sigma} g_{\nu\rho} - g_{\lambda\nu} g_{\sigma\rho} \right)} \qquad (15)$$

Use Bianchi identitities to say something about the position dependence of $R^{\lambda}{}_{\lambda}$

BI $\quad R_{\lambda\mu\nu\kappa;\eta} + R_{\lambda\mu\eta\nu;\kappa} + R_{\lambda\mu\kappa\eta;\nu} = 0 \quad \Big| \cdot g^{\lambda\nu}$

$$R_{\mu\kappa;\eta} - R_{\mu\eta;\kappa} + R^{\nu}{}_{\mu\kappa\eta;\nu} = 0 \quad \Big| \cdot g^{\mu\kappa}$$

$$R_{;\eta} - R^{\mu}{}_{\eta;\mu} - R^{\nu}{}_{\eta;\nu} = 0$$

$$\left(R^{\mu}{}_{\eta} - \frac{1}{2} \delta^{\mu}_{\eta} R \right)_{;\mu} = 0 \quad \Rightarrow \quad \underline{\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right)_{;\mu} = 0}$$

$\Downarrow$ with $R^{\mu\nu} = \frac{1}{N} g^{\mu\nu} R$

$$\left(\frac{1}{N} \delta^{\mu}_{\eta} R - \frac{1}{2} \delta^{\mu}_{\eta} R \right)_{;\mu} = \left(\frac{1}{N} - \frac{1}{2} \right) R_{;\eta} = \underline{\left(\frac{1}{N} - \frac{1}{2} \right) R_{,\eta} = 0} \qquad R \text{ scalar!}$$

$$N \neq 2 \qquad \underline{R_{,\eta} = 0}$$

$\Rightarrow$ <u>maximally symmetric spaces have a constant curvature</u>

Maximally symmetric spaces - construction

maximally symm. spaces → essentially unique (specified by a curvature constant K & number of positive or negative eigenvalues of the metric)

⇒ consider a flat (N+1) dim space with metric given by

$$ds^2 = g_{AB}\, dx^A dx^B = C_{\mu\nu}\, dx^\mu dx^\nu + \frac{1}{K} dz^2 \quad (1) \qquad A = 1, \dots, N+1 \quad \mu, \nu = 1, \dots, N$$

where

$C_{\mu\nu}$ is a constant (N×N) matrix symmetric   K some constant

⇒ Embed a non-Euclidean N-dim space in a larger space by restricting the variables x^μ & z to the surface of a (pseudo)sphere:

$$K C_{\mu\nu} x^\mu x^\nu + z^2 = 1 \quad (2) \quad \Rightarrow \quad 2z\,dz + 2K C_{\mu\nu} x^\mu dx^\nu = 0$$

$$\Rightarrow (dz)^2 = \dots$$

$$ds^2 = C_{\mu\nu}\, dx^\mu dx^\nu + \frac{K \left(C_{\mu\nu}\, dx^\mu x^\nu \right)^2}{1 - K C_{\mu\nu} x^\mu x^\nu}$$

⇓

metric

$$g_{\mu\nu}(x) = C_{\mu\nu} + \frac{K}{1 - K C_{\mu\nu} x^\mu x^\nu} \left(C_{\mu\alpha} x^\alpha C_{\nu\sigma} x^\sigma \right)$$

(1) and (2) invariant under $x'^A = R^A{}_B x^B$ $g_{AB} R^A{}_C R^B{}_D = g_{CD}$ ⇕ $\frac{1}{2}N(N+1)$ parameter

- $x'^\mu = R^\mu{}_\nu x^\nu$ with $C_{\mu\nu} R^\mu{}_\rho R^\nu{}_\sigma = C_{\rho\sigma}$
  $\frac{1}{2}N(N-1)$ parameter (isotropy)
- $x'^\mu = R^\mu{}_A x^A = x^\mu + a^\mu \cdot [\sim]$ (homogeneity: origin is taken into any other point)
  N parameter

⇒ $\frac{1}{2}N(N+1)$ parameter ⇒ max symmetric

Spaces with maximally symmetric subspaces

suitable coordinate system (N-dim) v^a (N−M) dim $\underbrace{u^i \text{ M-dim}}_{\text{max. symm. subspace}}$

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu = g_{ab}(v)\, dv^a dv^b + f(v)\, \tilde{g}_{ij}(u)\, du^i du^j$$

$\tilde{g}_{ij}$ ≙ metric of max. symm. subspace

Many cases of practical importance → max. symmetric spaces are SPACES, as opposed to space-times, so all eigenvalues of the submatrix $\tilde{g}_{ij}$ are positive!

Take for $K \neq 0$ $\quad C_{ij}$ as $\frac{1}{|K|}\delta_{ij}$

$$d\vec{\ell}^2 = \left(\frac{1}{|K|}\delta_{ij} + \frac{K}{(1 - \frac{K}{|K|}\delta_{km}x^k x^m)} \cdot \frac{1}{|K|^2} x_i x_j \right) dx^i dx^j$$

$\Downarrow$

$K > 0$ $\quad d\vec{\ell}^2 = \frac{1}{K}\left(d\vec{x}^2 + \frac{(\vec{x}\cdot d\vec{x})^2}{1-\vec{x}^2} \right)$

$K < 0$ $\quad d\vec{\ell}^2 = \frac{1}{K}\left(d\vec{x}^2 - \frac{(\vec{x}\, d\vec{x})^2}{1+\vec{x}^2} \right)$

$K = 0$ $\quad C_{ij} = \delta_{ij} \qquad d\vec{\ell}^2 = d\vec{x}^2$

$\Downarrow$

$$ds^2 = g_{ab}(v)\, dv^a dv^b + f(v)\left\{ d\vec{u}^2 + \frac{k(\vec{u}\, d\vec{u})^2}{1-k\vec{u}^2} \right\}$$

$k = +1$ for $K > 0$
$k = -1$ $\quad K < 0$
$k = 0$ $\quad K = 0$

⇒ Spherically symmetric homogeneous space-time

$N = 4$ & 3dim maximally symmetric subspace

$$ds^2 = g(v)\, dv^2 - f(v)\left\{ d\vec{u}^2 + \frac{k(\vec{u}\, d\vec{u})}{1-k\vec{u}^2} \right\}$$

new coordinates $\quad t = \int \sqrt{g(v)}\, dv$

$u^1 = r\sin\theta\cos\varphi$
$u^2 = r\sin\theta\sin\varphi$
$u^3 = r\cos\theta$

$$ds^2 = dt^2 - R^2(t)\left\{ \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 \right\}$$