$\boxed{\text{Virial Theorem}}$ - statistical theorem

system of masses m_j at positions $\vec{x}_j$, force on $m_j \hat{=} \vec{F}_j$

$$\frac{d}{dt}\sum_j \vec{p}_j \vec{x}_j = \sum_j \vec{p}_j \dot{\vec{x}}_j + \sum_j \dot{\vec{p}}_j \vec{x}_j = 2E_{kin} + \sum_j \vec{F}_j \vec{x}_j$$

$$\vec{p}_j \cdot \dot{\vec{x}}_j = \vec{p}_j \cdot \vec{v}_j = m \cdot \vec{v}_j^{\,2} = 2E_{kin}(j) \qquad E_{kin} = \sum_j E_{kin}(j)$$

time average of both sides

$$\frac{1}{\tau}\int_0^\tau \frac{d}{dt}\sum_j \vec{p}_j \cdot \vec{x}_j = \left\langle 2E_{kin} + \sum_j \vec{F}_j \cdot \vec{x}_j \right\rangle$$

bound system : each member of the assembly remains a member for all times

$\vec{x}_j$. finite values

$\vec{p}_j$ ... remains finite $\Rightarrow \sum_j \vec{p}_j \cdot \vec{x}_j$ remains finite

$$\Rightarrow \lim_{\tau \uparrow \infty} \frac{1}{\tau}\int_0^\tau \frac{d}{dt}\left(\sum_j \vec{p}_j \vec{x}_j\right) dt \to 0$$

$$\Downarrow$$

$$\langle 2E_{kin}\rangle + \left\langle \sum_j \vec{F}_j \vec{x}_j \right\rangle = 0$$

if the force is derivable from a potential $\vec{F}_j = -\vec{\nabla} E_{pot}(x_j)$

$$\Rightarrow \langle 2E_{kin}\rangle - \left\langle \underbrace{\sum_j \vec{\nabla} E_{pot}(\vec{x}_j) \cdot \vec{x}_j} \right\rangle = 0$$

$$\text{if } V(x) \sim x^n \qquad \frac{\partial E_{pot}(\vec{x}_j)}{\partial x_j} \cdot x_j = n \cdot E_{pot}(j) \qquad \sum_j E_{pot}(j) = E_{pot}$$

$$\Downarrow$$

$$2\langle E_{kin}\rangle = n \langle E_{pot}\rangle \qquad \langle E_{kin}\rangle = \frac{n}{2}\langle E_{pot}\rangle$$

$$\langle E_{kin}\rangle + \langle E_{pot}\rangle = \langle E_{pot}\rangle\left(1 + \frac{n}{2}\right) = \langle E_{tot}\rangle \Rightarrow n > -2$$

gravitation, electrostatic $\hat{=}$ $n = -1$

$$\Downarrow$$

$$\langle E_{kin}\rangle = \frac{1}{2}\langle E_{pot}\rangle$$