Cosmology and particle physics

Lecture notes

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Lecture 8   The thermal universe - part IV

In this lecture we discuss the Boltzmann equation that allows one to describe the evolution of processes in our universe that are not in equilibrium. Then we discuss the formation of light elements during big bang nucleosynthesis and the recombination of electrons and protons into neutral hydrogen.

1   The Boltzmann Equation

The number density in the absence of interactions (or in equilibrium) scales like the inverse volume, i.e. like a^{-3}, since it is a density. This means that it satisfies the equation

$$\frac{dn}{dt} + 3\frac{\dot{a}}{a}n = \frac{dn}{dt} + 3Hn = \frac{1}{a^3}\frac{d(na^3)}{dt} = 0\,. \tag{1}$$

As we discussed in the previous lectures, there are ample interactions in which two particle interact and become two new particles. These can be schematically written as

$$1 + 2 \leftrightarrow 3 + 4\,, \tag{2}$$

which means that particle 1 and 2 annihilate and become particles 3 and 4 (and vice versa). Such interactions together with decays of single particles are the most relevant processes in the early universe since the interaction of three or more particles is much more unlikely because these three or more particles would have to be all very close at the same time.

The Boltzmann equation describes the evolution of the number density n_1 of for example particle 1 in the presence of interactions. Here we focus on the interaction (2), in which case the Boltzmann equation is given by

$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = - < \sigma v > n_1 n_2 + c\, n_3 n_4\,, \tag{3}$$

where the first term describes the reduction of n_1 due to annihilation of particles 1 with 2, while the second term describes the production of 1 particles (and 2 particles) due to the annihilation of 3 and 4 particles. The free parameter c can be related to the thermally averaged cross-section $< \sigma v >$: We know from equation 1 that the right-hand-side of equation 3 has to vanish in thermal equilibrium, i.e. for $n_i = n_i^{eq}$. This gives

$$c = \frac{n_1^{eq} n_2^{eq}}{n_3^{eq} n_4^{eq}} < \sigma v > \,. \tag{4}$$

The Boltzmann equation then becomes

$$\frac{1}{a^3}\frac{d(n_1 a^3)}{dt} = - < \sigma v > \left(n_1 n_2 - \frac{n_1^{eq} n_2^{eq}}{n_3^{eq} n_4^{eq}} n_3 n_4 \right) . \tag{5}$$

This can be rewritten as

$$\frac{d\log(n_1 a^3)}{d\log(a)} = -\frac{\Gamma_1}{H}\left(1 - \frac{n_1^{eq} n_2^{eq}}{n_3^{eq} n_4^{eq}} \frac{n_3 n_4}{n_1 n_2} \right) , \tag{6}$$

where $\Gamma_1 = n_2 < \sigma v >$. The above equation determines the evolution of the number density for particles species 1 as a function of $a(t)$. Since $a(t)$ grows with t in our universe we can essentially think of the above Boltzmann equation as determining the evolution of species 1 with time. We see that Γ_1/H plays a crucial role in determining the evolution of $n_1 a^3$. If the interaction rate Γ_1 becomes small compared to the Hubble rate H, we have a freeze out and the number density of n_1 scales like a constant times a^{-3}.

To describe the evolution of all the particles in our early universe one has to solve simultaneously all the corresponding *coupled* Boltzmann equations. This is of course only possible numerically and goes beyond what we will discuss in class. Here we will focus on a few simple interesting cases that we can discuss more or less analytically and using the equilibrium results from the previous lectures. We will henceforth drop the superscript eq and just write n_i for the number densities in equilibrium.

Chemical potentials

Before we discuss big bang nucleosynthesis it is useful to review the effect of a non-zero chemical potential. In the phase space distribution function (see for example equation (1) in the lecture 7 notes) a non-zero chemical potential leads to

$$f_\pm(p) = \frac{1}{e^{(E(p)-\mu)/T} \pm 1} . \tag{7}$$

While again each particle can have a different chemical potential, chemical equilibrium, which is reached via interactions, leads to relations between the chemical potentials. For example interaction like the ones in equation (2) lead to

$$\mu_1 + \mu_2 = \mu_3 + \mu_4 . \tag{8}$$

Non-zero chemical potentials will modify the expression for, for example, the number density, so that for non-relativistic particles in equilibrium it is given by

$$n = g\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} e^{\frac{\mu - m}{T}} . \tag{9}$$

However, if we take ratios of number densities in which the chemical potential cancels due to equation (8), then we don't really need the values of the chemical potentials.

Note, that photons can interact with electrons via a double Compton scattering

$$e^- + \gamma \leftrightarrow e^- + \gamma + \gamma , \tag{10}$$

which leads to $\mu_\gamma = 0$.

2 Big bang nucleosynthesis

Big bang nucleosynthesis refers to the formation of atomic nuclei during the cooling of our early universe. Recall that after the QCD phase transition around $150MeV$ quarks form colorless bound states that include protons and neutrons. During the continuous cooling of our universe, the number densities of these non-relativistic baryons is exponentially decaying until, due to the initial antisymmetry between baryons and anti-baryons, we are left with a residual amount of baryonic matter in the form of protons and neutrons and heavier nuclei. The protons and neutrons can bind via the strong force into atomic nuclei and via the weak force neutrons and protons can convert into each other. All these processes are initially in equilibrium and we want to understand with which relic abundance of nuclei we are left, once these processes drop out of equilibrium due to the cooling of our universe.

The two reasons why we can actually do that without solving many coupled Boltzmann equations are firstly that essentially no elements heavier than Helium are created during big bang nucleosynthesis, so that we can just focus our attention on Hydrogen and Helium and secondly that initially we have only neutrons and protons in equilibrium without any relevant amount of heavier nuclei.

2.1 Protons and neutrons

At temperatures above $1MeV$ protons and neutrons are in equilibrium due to weak interactions of the form

$$n + \nu_e \leftrightarrow p^+ + e^- \,. \tag{11}$$

One can argue that the chemical potentials for electrons and neutrinos are negligible small, so that equation (8) tells us that $\mu_p = \mu_n$. Taking the ratio of the proton and neutron number densities, the chemical potential then simply cancels (see equation (9)) and we find

$$\frac{n_n}{n_p} = \left(\frac{m_n}{m_p}\right)^{\frac{3}{2}} e^{-\frac{m_n - m_p}{T}} \,. \tag{12}$$

Recalling the proton and neutron masses $m_p = 938.27MeV$ and $m_n = 939.57MeV$, we see that their ratio is very close to 1 and their difference is $m_n - m_p = 1.3MeV$. So at large temperatures $T \gg 1MeV$ we have the same number of neutrons and protons, while at energies below $T \sim 1MeV$, the ratio of neutron and proton number densities is exponentially decaying. However, as we have seen last time when we discussed neutrinos, processes that involve the weak interactions like the one in equation (11) will become irrelevant at energies below roughly $1MeV$, since $\Gamma/H \approx 1$ for $T \approx 1MeV$ (see equation (14) in the lecture 7 notes). Actually a more careful analysis reveals that the weak interactions become irrelevant at $T \approx .8MeV$ which leads to

$$\frac{n_n}{n_p} = \left(\frac{m_n}{m_p}\right)^{\frac{3}{2}} e^{-\frac{m_n - m_p}{T}} \approx e^{-\frac{1.3MeV}{.8MeV}} \approx .2 \,. \tag{13}$$

Once the temperature drops further the finite lifetime of the neutron becomes important. In particular, a free neutron can decay via

$$n \to p^+ + e^- + \bar{\nu}_e \,, \tag{14}$$

which leads to an exponential decay of the neutron number density

$$\frac{n_n}{n_p} \to \frac{n_n}{n_p} e^{-\frac{t}{886s}} \approx .2e^{-\frac{t}{886s}}\,, \tag{15}$$

where we used that the mean lifetime of a free neutron is $886s$. The decay of the neutrons stops once they are bound into nuclei which happens around $t \approx 330s$ which leads to

$$\left.\frac{n_n}{n_p}\right|_{t\approx 330s} \approx .14\,. \tag{16}$$

2.2 Heavier nuclei

Let us study a process that involves the production of the lightest nucleus that is not just a proton, i.e. deuterium. One neutron and one proton can form deuterium (and a photon):

$$n + p^+ \leftrightarrow D^+ + \gamma\,. \tag{17}$$

As we argued above, the photon's chemical potential vanishes so that the chemical potentials cancel in the following ratios

$$\frac{n_D}{n_n n_p} = \frac{3}{4}\left(\frac{2\pi}{T}\frac{m_D}{m_n m_p}\right)^{\frac{3}{2}} e^{-\frac{m_D - m_n - m_p}{T}}\,, \tag{18}$$

where we used $g_n = g_p = 2$ and $g_D = 3$. The ratio between the masses is approximately $2/m_p$, however, the difference in the mass of the deuterium and its two constitutions is the binding energy $m_N + m_p - m_D \approx 2.2MeV$. At energies well below the proton and neutron masses, i.e. at $T \ll 1GeV$, the number densities of protons and neutrons are not exponentially decaying anymore but are determined by the non-zero baryon number in our universe, i.e. by equation (12) in the lecture 5 notes:

$$n_p \sim n_n \sim n_b \sim 10^{-9} n_\gamma = 10^{-9}\frac{2\zeta(3)}{\pi^2}T^3\,, \tag{19}$$

where we used equation (11) in the lecture 6 notes for the photon number density. Using this in equation (18), we get

$$\frac{n_D}{n_p} \approx 8\left(\frac{T}{m_p}\right)^{\frac{3}{2}} e^{\frac{2.2MeV}{T}} 10^{-9}\,. \tag{20}$$

This implies that for $T = 1MeV$, we have $n_D/n_p \approx 10^{-12}$ and for roughly $T \approx .066MeV$ we have $n_D/n_p \approx 1$. This means that at temperatures above $T \approx .1MeV$ the deuterium abundance is negligible and the same is true for even heavier nuclei.

2.3 Nucleosynthesis

Now we have all pieces in place and can discuss the creation of nuclei that are not just a proton. Our starting point are protons and neutrons. As we mentioned before, processes involving more than two particles are very rare so that the initial process must be the formation of deuterium from one proton and one neutron as shown in equation (17). Only once deuterium is formed, which as we saw above happens around $T \approx .66MeV$, can Helium be produced via

$$\begin{aligned} D + p^+ &\leftrightarrow {}^3He + \gamma\,, \\ D +^3 He &\leftrightarrow {}^4He + p^+\,. \end{aligned} \tag{21}$$

The binding energy of ^4He, B_{He}, is larger than that of deuterium B_D. This leads to an enhancement of the number density of Helium compared to that of deuterium

$$\frac{n_{He}}{n_D} \propto e^{\frac{B_{He}-B_D}{T}}\,. \tag{22}$$

This is similar to equation (20), where deuterium is favored at low temperatures, except that here we don't have a suppression factor. This means that helium is almost immediately produced after deuterium and that all neutrons end up in ^{4}He nuclei. Since each ^{4}He atoms contains two neutrons, this allows us to easily determine the fraction of helium to hydrogen in our universe

$$\frac{n_{He}}{n_H} = \frac{n_{He}}{n_p} = \frac{\frac{1}{2}n_n}{n_p} = 7\%\,. \tag{23}$$

This answer is very close to a full numerical analysis that solves all the coupled Boltzmann equations and which gives something like $6.2\% \approx \frac{1}{16}$. Since the mass of an Helium nucleus is roughly four times as large as the proton mass, we find that roughly one fourth of the mass of ordinary matter in our early universe is in the form of Helium and the rest in the form of Hydrogen. This perfectly agrees with observations and is one of the great successes of big bang nucleosynthesis and shown in figure 1.

Beyond Helium You probably wonder why heavier atomic nuclei don't form during big bang nucleosynthesis (and how they appeared in our universe). The reason that they aren't formed from protons, neutrons, deuterium and helium is the following: As we have seen above, before helium can be formed, protons and neutrons need to first combine to form a substantial amount of deuterium. During this time the universe keeps cooling and the nuclei loose part of their kinetic energy, which makes it harder to overcome the Coloumb barrier (i.e. to bring together two positively charge nuclei). More importantly, once a large amount of ^4He is formed, these can only combine to form ^8Be which is unstable and decays faster than it can be formed. Very small amounts of Tritium and ^3He that are also created during big bang nucleosynthesis can combine with ^4He to form ^7Li of which we observe tiny amounts today.[1] So big bang nucleosynthesis produces only very light elements. As briefly mentioned last time, the heavier elements that we see today in our universe and that we are made of are created in the first stars through nuclear fusions. Once the heaviest of these stars explode in supernovae, these heavy elements are released into the stellar medium and can become part of second generation stars and form planets.

[1] Actually we observe slightly more Lithium than theoretical predicted (see figure 1) which might require small modifications of big bang nucleosynthesis.

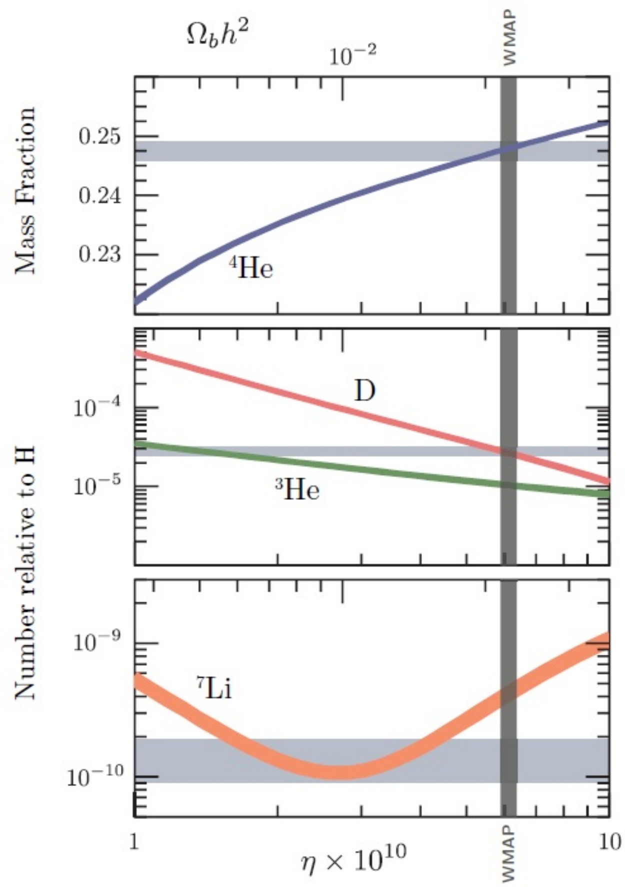

Figure 1: The mass fraction of several light nuclei as theoretical predicted (colored band) and measured by WMAP (grey band).

3 Recombination and photon decoupling

After nuclei are formed around $T \approx .06 MeV$, which corresponds roughly to a time of 3 minutes after the big bang, our universe contains a soup of positively charged nuclei, free electrons and photons (as well as decoupled neutrinos). During the further expansion and cooling of our universe the energy density of radiation decays like a^{-4}, while the energy density in the non-relativistic matter decays like a^{-3}. Roughly 60,000 years after the big bang the energy densities in radiation and matter are equal and our universe enters its matter dominated era. Another 200,000 years later electrons and nuclei start to form neutral atoms, during a period that is usually called recombination.[2] Once the recombination ends and the universe essentially consists of neutral atoms, the photons in the cosmic microwave background can stream freely until today and tell us about the universe 380,000 years after the big bang as well as much earlier times.

[2]A more accurate name would be *combination* since they have never been combined before.

3.1 The Saha equation

The process that keeps electrons, protons and photons in equilibrium during the first 200,000 years after the big bang is

$$e^- + p^+ \leftrightarrow H + \gamma\,. \tag{24}$$

At a temperature of $T \approx 1eV$ all particles except the photons are non-relativistic and in (chemical and thermal) equilibrium due to the above process. Recalling that the photon chemical potential vanishes, we can look at the following ratio in which the chemical potentials cancel[3]

$$\frac{n_H}{n_p n_e} = \left(\frac{m_H}{m_e m_p}\frac{2\pi}{T}\right)^{\frac{3}{2}} e^{\frac{m_p+m_e-m_H}{T}}\,, \tag{25}$$

where we used that $g_H = 4 = g_e g_p$. Using that $m_p \approx m_H$ and that the binding energy of hydrogen is $m_p + m_e - m_H = 13.6eV$ and the fact that our universe is electrically neutral which implies $n_e = n_p$, we find

$$\frac{n_H}{n_e^2} = \left(\frac{2\pi}{m_e T}\right)^{\frac{3}{2}} e^{\frac{13.6eV}{T}}\,, \tag{26}$$

Next we define the free electron fraction as the ratio of free electrons to baryons

$$X_e \equiv \frac{n_e}{n_b}\,. \tag{27}$$

As we have seen above in equation (23), more than 90% of the baryon number is due to the protons, that can be in the form of positively charged nuclei n_p or in the form of neutral hydrogen n_H, so that

$$n_b \approx n_p + n_H = n_e + n_H \approx 10^{-9}\frac{2\zeta(3)}{\pi^2}T^3\,, \tag{28}$$

where we used equation (19). From the definition in equation (27) we then find

$$\frac{1-X_e}{X_e^2} = \frac{(n_p+n_H)-n_e}{n_e^2}(n_p+n_H) = \frac{n_H}{n_e^2}n_b\,. \tag{29}$$

Using this in equation (26) we find the Saha equation

$$\frac{1-X_e}{X_e^2} = 10^{-9}\frac{2\zeta(3)}{\pi^2}\left(\frac{2\pi T}{m_e}\right)^{\frac{3}{2}} e^{\frac{13.6eV}{T}}\,. \tag{30}$$

3.2 Recombination

The Saha equation allows us to get an estimate for the energies at which recombination happened. Taking the onset of recombination as the temperature $T_{beginning}$ at which $X_e = .9$ and the end of recombination as the temperature T_{end} at which $X_e = .1$ we find from equation (30) that $T_{beginning} \approx .35eV$ and $T_{end} \approx .30eV$. The reason that these results are so much smaller than the 13.6eV binding energy is that there are many, many more photons than baryons and that the black body spectrum of the photons has a tail of high energy photons that keep the Hydrogen ionized until the average temperature of the photon bath is well below the binding energy of Hydrogen.

[3] Here we use n_H to denote the neutral hydrogen only so that $n_p \neq n_H$.

3.3 Photon decoupling

The so called time of last scattering at which the electrons and photons scatter for the last time via Thompson scattering

$$e^- + \gamma \leftrightarrow e^- + \gamma \,, \tag{31}$$

is actually happening even later around a time when $X_e \approx .01$. We see this as follows: The cross-section for Thompson scattering is $\sigma_T \approx 2 \times 10^{-3} MeV^{-2}$ and the corresponding interaction rate is given by

$$\Gamma_T \approx n_e \sigma_T = n_b X_e \sigma_T \approx 10^{-9} \frac{2\zeta(3)}{\pi^2} T^3 X_e \sigma_T \,. \tag{32}$$

In order to determine the temperature at decoupling T_{dec} we have to check when the above interaction rate is of the same size as the Hubble expansion rate. During matter domination the Hubble rate is given by [4]

$$H = H_0 \sqrt{\Omega_{m,0}} \left(\frac{T}{T_0} \right)^{\frac{3}{2}} . \tag{33}$$

This implies

$$\Gamma(T_D) = H(T_D) \qquad \Leftrightarrow \qquad T_D^{\frac{3}{2}} X_e(T_D) = 10^9 \frac{\pi^2}{2\zeta(3)} \frac{H_0 \sqrt{\Omega_{m,0}}}{\sigma_T T_0^{\frac{3}{2}}} . \tag{34}$$

We can numerically solve this equation and find $T_D \approx .26eV$ and $X_e(T_D) \approx .003$. The temperature $T_D \approx .26eV$ corresponds to a time of 380,000 years after the big bang and a redshift of $z \approx 1100$.

[4] This follows from the first Friedmann equation

$$H^2 = H_0^2 \frac{8\pi G}{3H_0^2} \rho_m = H_0^2 \Omega_{m,0} \left(\frac{a_0}{a(t)} \right)^3$$

and the fact that $a(t) \sim 1/T$.