Cosmology and particle physics

Lecture notes

Timm Wrase

Lecture 10 Inflation - part II

Last time we discussed how a period of inflation can solve several problems that we encounter in our very early universe. In this lecture we are studying the relevant equations for inflation as well as a few exemplary models.

1 A scalar field

As we have seen last time, a new scalar field, called the inflaton, can lead to a temporary phase of inflation. To make this precise we vary the action for the scalar field

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) \right) , \tag{1}$$

with respect to the scalar field to derive its equation of motion. In order to do this recall that the FRW metric takes the form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a(t)^2 \gamma_{ij} dx^i dx^j \equiv -dt^2 + a(t)^2 \left(dx_i^2 + K \frac{x_i^2 dx_i^2}{1 - K x_i^2} \right) . \tag{2}$$

This then leads to

$$\begin{aligned}
\delta S &= \int d^4x a(t)^3 \left(-g^{\mu\nu} \partial_\mu \phi \partial_\nu \delta\phi - V'(\phi) \delta\phi \right) \\
&= \int d^4x \left[\partial_\nu \left(a(t)^3 g^{\mu\nu} \partial_\mu \phi \right) - a(t)^3 V'(\phi) \right] \delta\phi \\
&= \int d^4x \left[-\partial_t \left(a(t)^3 \partial_t \phi \right) + \partial_i \left(a(t) \gamma^{ij} \partial_j \phi \right) - a(t)^3 V'(\phi) \right] \delta\phi \\
&= \int d^4x \left[-3\dot{a}(t) a(t)^2 \dot{\phi} - a(t)^3 \ddot{\phi} + a(t) \nabla^2 \phi - a(t)^3 V'(\phi) \right] \delta\phi \\
&= \int d^4x \left(-a(t)^3 \right) \left[\ddot{\phi} + 3 \frac{\dot{a}(t)}{a(t)} \dot{\phi} - \frac{\nabla^2 \phi}{a(t)^2} + V'(\phi) \right] \delta\phi \, ,
\end{aligned} \tag{3}$$

where we used the short-hand notation $V'(\phi) = \partial_\phi V(\phi)$. So the equation of motion for a scalar field in an FRW universe is given by

$$\ddot{\phi} + 3H\dot{\phi} - \frac{\nabla^2 \phi}{a(t)^2} + V'(\phi) = 0 \, . \tag{4}$$

Once inflation starts, $a(t)$ grows exponentially so that the term with the spatial derivatives of ϕ quickly becomes unimportant and the above equation reduces to

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0 \, . \tag{5}$$

We can also simply derive this equation for a spatially homogeneous scalar field from the continuity equation $\dot{\rho} + 3H(\rho_\phi + P_\phi) = 0$ by plugging in the energy density and pressure for a homogeneous scalar field

$$\begin{aligned} \rho_\phi &= \frac{1}{2}\dot{\phi}^2 + V(\phi)\,, & (6) \\ P_\phi &= \frac{1}{2}\dot{\phi}^2 - V(\phi)\,. & (7) \end{aligned}$$

Note that equation (5) that describes the evolution of a homogeneous scalar field is very similar to a harmonic oscillator. [1] The derivative of the scalar potential acts like a driving force and the expansion of our universe leads to the friction term $3H\dot{\phi}$.

2 Slow-roll inflation

As we discussed last time, the successful models of inflation have a very flat potential along which the scalar field rolls down towards a minimum of the potential. If the scalar fields rolls so slow, that we can neglect $\dot{\phi}^2$ compared to the potential value $V(\phi)$, then the scalar field behaves approximately like a cosmological constant, $\rho_\phi \approx -P_\phi$, and the universe undergoes a period of exponential expansion. This is called *slow-roll inflation* since the inflaton is rolling down the potential very slowly. Because matter and radiation will be diluted away due to this exponential expansion, we can neglect any other source of energy density and simply focus on the scalar field. We will get back to setting the initial conditions for our hot big bang scenario next time, when we discuss the end of inflation and the reheating of our universe.

There are two small (dimensionless) parameters that allow us to make the condition of a slowly rolling scalar field more precise. Recall that the first Friedmann equation sourced by a homogeneous scalar field takes the form

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = H^2 = \frac{8\pi G}{3}\rho_\phi = \frac{1}{3M_P^2}\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)\,. \tag{8}$$

So we see that for $\dot{\phi}^2 \ll V(\phi)$ we have a nearly constant potential $V(\phi)$ value since the scalar field is only changing very slowly in time. This then implies a nearly constant Hubble parameter, so that it is very useful to introduce the dimensionless slow-roll parameter

$$\epsilon \equiv -\frac{\dot{H}}{H^2}\,. \tag{9}$$

Note that during a period of almost exponential expansion $\dot{H} < 0$ so that $\epsilon > 0$ (see equation (15) below). A period of inflation requires $\epsilon \ll 1$. Since we need inflation to last sufficiently long, we need ϵ not to change that quickly which is captured by the second dimensionless slow-roll parameter

$$\eta \equiv \frac{\dot{\epsilon}}{H\epsilon}\,. \tag{10}$$

[1] If we choose $V(\phi) = \frac{1}{2}m^2\phi^2$, then the equation of motion is identical to a harmonic oscillator with friction.

This parameter keeps track of the relative change $\dot{\epsilon}/\epsilon$ per Hubble time and also needs to be small for an extended period of inflation.

Last time we have already used the idea of e-folds that measure the number of exponential expansions (to basis e) of our universe. We already defined the total number of e-folds as N_e but it is often more useful to measure time in terms of the number of e-folds. To this end we define

$$dN \equiv d\ln(a) = Hdt\,. \tag{11}$$

The total number of e-folds N_e is then given by

$$N_e = \int_{a_i}^{a_f} d\ln a = \ln\left(\frac{a_f}{a_i}\right) = \int_{t_i}^{t_f} Hdt \approx H_{inf}(t_f - t_i)\,, \tag{12}$$

where H_{inf} is the approximately constant Hubble parameter during inflation.

The second Friedmann equation

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}(\rho(t) + 3P(t)) = -\frac{1}{6M_P^2}(\rho(t) + 3P(t))\,, \tag{13}$$

in the presence of only a homogeneous scalar field gives

$$\dot{H} + H^2 = -\frac{1}{3M_P^2}(\dot{\phi}^2 - V(\phi))\,. \tag{14}$$

Using equation (8) we then get

$$\dot{H} = -\frac{\dot{\phi}^2}{2M_P^2}\,. \tag{15}$$

Plugging this into the definition of ϵ we find

$$\epsilon = \frac{\dot{\phi}^2}{2M_P^2 H^2}\,. \tag{16}$$

Taking the time derivative gives

$$\dot{\epsilon} = \frac{\ddot{\phi}\dot{\phi}}{M_P^2 H^2} - \frac{\dot{\phi}^2 \dot{H}}{M_P^2 H^3}\,. \tag{17}$$

We can use this to rewrite η

$$\eta \equiv \frac{\dot{\epsilon}}{H\epsilon} = \left(\frac{\ddot{\phi}\dot{\phi}}{M_P^2 H^2} - \frac{\dot{\phi}^2 \dot{H}}{M_P^2 H^3}\right)\frac{2M_P^2 H}{\dot{\phi}^2} = 2\frac{\ddot{\phi}}{\dot{\phi}H} - 2\frac{\dot{H}}{H^2} = 2\frac{\ddot{\phi}}{\dot{\phi}H} + 2\epsilon\,. \tag{18}$$

2.1 The slow-roll equations

So far we have not really made any approximations but since during inflation ϵ and η are very small we can calculate them and everything else to leading order to get rather simple expressions. For example, a small $\epsilon \ll 1$ (see eq. (16)) implies that $\dot\phi^2/(2M_P^2) \ll H^2$. It then follows from equation (8) that during slow-roll inflation

$$H^2 \approx \frac{V}{3M_P^2}\,. \tag{19}$$

This means that the Hubble constant during inflation is set by the value of the scalar potential. Since the scalar field is slowly rolling the value of the potential is only changing very slowly and therefore the Hubble constant is approximately constant during inflation. Similarly, a small $|\eta|$ and ϵ implies due to equation (18) that $\ddot\phi \ll \dot\phi H$. It then follows from equation (5) that

$$3H\dot\phi \approx -V'(\phi)\,. \tag{20}$$

Taking the time derivative of the above equation we get

$$3\dot H\dot\phi + 3H\ddot\phi \approx -\dot\phi V''(\phi)\,. \tag{21}$$

Using the equations (19), (20) and (21), we find the following approximate expression for ϵ and η

$$\begin{aligned}
\epsilon &= \frac{\dot\phi^2}{2M_P^2H^2} \approx \frac{(V')^2}{18M_P^2H^4} = \frac{M_P^2}{2}\left(\frac{V'}{V}\right)^2\,,\\
\eta &= 2\frac{\ddot\phi}{\dot\phi H} + 2\epsilon \approx \frac{2}{\dot\phi H}\left(-\frac{\dot\phi V''}{3H} - \frac{\dot H\dot\phi}{H}\right) + 2\epsilon = -2M_P^2\frac{V''}{V} + 2M_P^2\left(\frac{V'}{V}\right)^2\,.
\end{aligned} \tag{22}$$

So we see that we can express ϵ and η entirely in terms of the scalar potential. It is convenient to introduce the slow-roll parameters ϵ_V and η_V that are defined by

$$\epsilon_V \equiv \frac{M_P^2}{2}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \approx \epsilon\,, \tag{23}$$

$$\eta_V \equiv M_P^2\frac{V''(\phi)}{V(\phi)} \approx -\frac{1}{2}\eta + 2\epsilon\,. \tag{24}$$

The smallness of ϵ_V is equivalent to the condition that the first derivative of the potential is small compared to the value of the potential, while the smallness of $|\eta_V|$ is equivalent to the smallness of the second derivative of the potential. It follows from the equations (22), that the slow-roll conditions $\epsilon, \eta \ll 1$ are equivalent to $\epsilon_V, \eta_V \ll 1$.

We can also express the number of e-folds of inflation in terms of the slow-roll parameter ϵ_V. Using equation (16) we can rewrite

$$H dt = \frac{H}{\dot\phi}d\phi = \frac{1}{\sqrt{2\epsilon}}\frac{|d\phi|}{M_P} \approx \frac{1}{\sqrt{2\epsilon_V}}\frac{|d\phi|}{M_P}\,. \tag{25}$$

Now we use this in the definition of the number of e-folds given in equation (12) to get

$$N_e = \int_{t_i}^{t_f} H dt \approx \int_{\phi_i}^{\phi_f} \frac{1}{\sqrt{2\epsilon_V}}\frac{|d\phi|}{M_P} = \frac{1}{M_P^2}\left|\int_{\phi_i}^{\phi_f}\frac{V(\phi)}{V'(\phi)}d\phi\right|\,. \tag{26}$$

3 Examples of inflationary models

3.1 Natural inflation

After discussing all the relevant equations, let us now discuss a concrete model of inflation that is called natural inflation. In this model the inflaton field has a discrete shift symmetry $\phi \to \phi + 2\pi f$. The potential that respects this shift symmetry is given by

$$V(\phi) = \lambda^4 \left[1 + \cos\left(\frac{\phi}{f}\right)\right] . \tag{27}$$

Here we have set the minimum value of the potential equal to zero since the current cosmological constant is so small that it would not matter for the period of inflation, if we add to this potential a constant that is $10^{-120} M_P^4$ or not. One period of the potential is shown in figure (1).

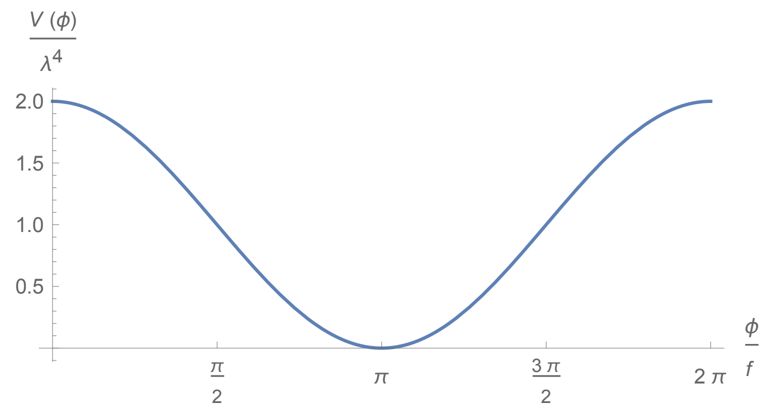

Figure 1: The scalar potential for natural inflation.

Now we calculate the slow-roll parameters

$$\epsilon_V = \frac{M_P^2}{2f^2}\left(\frac{\sin\left(\frac{\phi}{f}\right)}{1+\cos\left(\frac{\phi}{f}\right)}\right)^2 , \tag{28}$$

$$\eta_V = -\frac{M_P^2}{f^2}\frac{\cos\left(\frac{\phi}{f}\right)}{1+\cos\left(\frac{\phi}{f}\right)} . \tag{29}$$

Since the scalar potential is zero (or very small) at the minimum of the potential, while the second derivative is large, we see that we can not satisfy $|\eta_V| \ll 1$ near the minimum of the potential. [2] Away from the minimum at $\phi = \pi f$, there is no enhancement of ϵ_V and η_V from the denominator but we don't have any substantial suppression from the nominator either,

[2]The second derivative of the scalar potential at the minimum determines the mass of the inflaton today. Since we don't observe any very light scalar fields, the second derivative at the minimum has to be much larger than the value of the potential.

since it is not possible to have $|\sin(\phi/f)| \ll 1$ and at the same time $|\cos(\phi/f)| \ll 1$. So the only way to get small ϵ_V and η_V in this model is by choosing $f \gg M_P$. This automatically suppresses both slow-roll parameters and allows for a period of slow-roll inflation.

We can calculate the number of e-folds in this model using equation (26)

$$N_e \approx \int_{\phi_i}^{\phi_f} \frac{1}{\sqrt{2\epsilon_V}} \frac{|d\phi|}{M_P} = \frac{f}{M_P^2} \int_{\phi_i}^{\phi_f} \left| \frac{1+\cos\left(\frac{\phi}{f}\right)}{\sin\left(\frac{\phi}{f}\right)} \right| |d\phi| = 2\frac{f^2}{M_P^2} \left| \log\left(\frac{\sin\left(\frac{\phi_f}{2f}\right)}{\sin\left(\frac{\phi_i}{2f}\right)} \right) \right| . \tag{30}$$

We see, probably as one would expect, that the number of e-folds diverges as one moves the starting point ϕ_i closer and closer to a maximum $\phi \in 2\pi n f$, $n \in \mathbb{Z}$.

The end of inflation is defined as the point at which one of the slow-roll parameters becomes equal to 1. Usually the expansion after that point can only add some order one number of e-folds. So this end of inflation is not the exact end of the exponential expansion but a rough guide. In our example ϵ_V and η_V are of the same order and which one is bigger depends on how close to the minimum we are. For simplicity we just focus on $\epsilon_V = 1$. One finds that

$$\epsilon_V(\phi_f) = 1 \qquad \Leftrightarrow \qquad \phi_f = f\left[\pi \pm \arctan\left(\frac{2\sqrt{c}}{c-1}\right)\right] , \tag{31}$$

where $c = \sqrt{2} f/M_P$. Since a period of slow-roll requires $f \gg M_P$, we have that $c \gg 1$ and we can approximate

$$\arctan\left(\frac{2\sqrt{c}}{c-1}\right) \approx \frac{2}{\sqrt{c}} = 2^{\frac{3}{4}} \sqrt{\frac{M_P}{f}} . \tag{32}$$

Now if we take for example $f = 100 M_P \gg M_P$, then we have $\phi_f \approx 2.97 f = 297 M_P$ (or $\phi_f \approx 2\pi f - 2.97 f \approx 3.31 f = 331 M_P$) and we find for the number of e-folds

$$N_e \approx 2 \times 10^4 \left| \log\left(\frac{\sin\left(\frac{\pi}{2} \pm \frac{2^{\frac{3}{4}}}{20}\right)}{\sin\left(\frac{\phi_i}{200 M_P}\right)} \right) \right| \approx -2 \times 10^4 \log\left| \sin\left(\frac{\phi_i}{200 M_P}\right) \right| - 71 . \tag{33}$$

If we want to get 60 e-folds, then we can numerically solve for ϕ_i and find $\phi_i \approx 2.91 f = 291 M_P$ (or $\phi_i \approx 2\pi f - 2.91 f \approx 337 M_P$). We see that in this example ϕ travels a distance in field space that is larger than M_P. Models of this type in which $\Delta\phi \equiv |\phi_i - \phi_f| \gtrsim M_P$ are called *large field models*. Such large field models are currently being tested by observations and are already highly constrained. Constructing these models in a controlled way provides many theoretical challenges (and ideally requires a full fledged theory of quantum gravity). In these large field models ϵ_V is much larger than in small field models. This in turn implies that large field models require less tuning of the scalar potential in order to get a period of inflation.

3.2 $m^2\phi^2$ inflation

As we also see from above, since $\phi_i \approx 2.91 f$ and $\phi_f \approx 2.97 f$, the inflaton field does not really explore much of the potential but stays very close to the minimum. In such cases we can just expand the potential around the minimum and find

$$V(\phi) = \lambda^4 \left[1 + \cos\left(\frac{\phi}{f}\right)\right] \approx \frac{\lambda^4}{2f^2}\left((\phi - \pi f)^2 + \mathcal{O}\left((\phi - \pi f)^4\right)\right) . \tag{34}$$

Keeping only the leading term, defining $m^2 = \lambda^4/f^2$ and shifting $\phi \to \phi + \pi f$ we find the potential

$$V(\phi) = \frac{1}{2}m^2\phi^2 . \tag{35}$$

This is arguably the simplest potential for large field inflation and it is very generic in the sense that for every massive scalar field the potential around the minimum is quadratic (since $m^2 \propto V''(\phi)$).

In this simple model the slow-roll parameters are

$$\epsilon_V = \eta_V = 2\left(\frac{M_P}{\phi}\right)^2 . \tag{36}$$

So as long as $\phi \gg M_P$ we have slow-roll inflation and inflation ends when $\phi \approx \sqrt{2}M_P$. The number of e-folds in this model is given by

$$N_e \approx \int_{\phi_i}^{\phi_f} \frac{1}{\sqrt{2\epsilon_V}}\frac{|d\phi|}{M_P} = \left|\int_{\phi_i}^{\sqrt{2}M_P} \frac{\phi d\phi}{2M_P^2}\right| = \frac{\phi_i^2}{4M_P^2} - \frac{1}{2} . \tag{37}$$

For sixty e-folds we need

$$\phi_i = 2M_P\sqrt{60.5} \approx 15.6M_P . \tag{38}$$

Plugging this into ϵ_V we find

$$\epsilon_V(\phi_i) = 2\left(\frac{M_P}{\phi_i}\right)^2 = \frac{1}{121} \approx .0083 . \tag{39}$$

The current bound from the Planck satellite from February 2015 just excluded this model at the 2σ confidence level by providing the upper bound $\epsilon_V \leq .0069$.

The above more general model of natural inflation has not yet been excluded but its parameter space is highly constraint and it is one of the models that will be excluded or confirmed in the near future.

3.3 Small field models of inflation

By definition small field models of inflation satisfy $\Delta\phi = |\phi_i - \phi_f| \ll M_P$. As an example let us again return to the model above and now focus on the region near a maximum, for example near $\phi = 0$. In this case we can expand all the equations around $\phi = 0$ and keep the leading terms. This gives us an example of so called hilltop inflation, in which the period of inflation happens near an unstable point of the potential.

The potential becomes

$$V(\phi) = \lambda^4 \left[2 - \frac{1}{2}\left(\frac{\phi}{f}\right)^2\right] , \tag{40}$$

which is unbounded from below but we of course only consider it as an approximation near the ϕ values for which inflation takes place. This potential is then only valid for $\phi/f \ll 1$ and does not need to be completed to a full cosine but can rather have arbitrary higher order corrections of $\mathcal{O}\left((\phi/f)^3\right)$.

The slow-roll parameters are

$$\begin{aligned} \epsilon_V &= \frac{M_P^2}{2}\left(\frac{\frac{\phi}{f^2}}{2}\right)^2 = \frac{M_P^2\phi^2}{8f^4}\,, \qquad (41)\\ \eta_V &= -\frac{M_P^2}{2f^2}\,. \qquad (42)\end{aligned}$$

We see that both are small as long as $M_P, \phi \ll f$. We assume that corrections to the potential will modify ϵ_V and η_V near $\phi \sim f/100$ and that inflation ends around this point. The number of e-folds is then given by

$$N_e \approx \int_{\phi_i}^{\phi_f} \frac{1}{\sqrt{2\epsilon_V}}\frac{|d\phi|}{M_P} = \frac{2f^2}{M_P^2}\int_{\phi_i}^{f/100}\frac{d\phi}{\phi} = \frac{2f^2}{M_P^2}\log\left(\frac{f}{100\,\phi_i}\right)\,. \qquad (43)$$

For example, for concreteness we can choose $f = 5M_P$ so that 60 e-folds require

$$N_e = 60 \approx 50\log\left(\frac{M_P}{20\,\phi_i}\right) \qquad \Leftrightarrow \qquad \phi_i \approx .015 M_P \ll M_P\,. \qquad (44)$$

This leads to

$$\epsilon_V(\phi_i) \approx 5\times 10^{-8} \qquad \Leftrightarrow \qquad M_P|V'(\phi_i)| \approx 2\times 10^{-4}\; V(\phi_i)\,, \qquad (45)$$

which is a larger fine tuning of the potential as one would require in large field models.