## Deriving the Friedmann equations from general relativity

The FRW metric in Cartesian coordinates is

$$ds^{2} = g_{\mu\nu}dx^{\mu}dx^{\nu} = -dt^{2} + g_{ij}dx^{i}dx^{j} = -dt^{2} + a(t)^{2}\left(dx_{i}^{2} + K\frac{x_{i}^{2}dx_{i}^{2}}{1 - Kx_{i}^{2}}\right),$$
 (1)

where Greek letters run over  $\mu, \nu, \ldots = 0, 1, 2, 3$  and latin letters  $i, j, \ldots = 1, 2, 3$ . The Christoffel symbol  $\Gamma^{\rho}_{\mu\nu}$  is given by

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma} \left[ \frac{\partial g_{\sigma\mu}}{\partial x^{\nu}} + \frac{\partial g_{\sigma\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\sigma}} \right] . \tag{2}$$

For the metric (1) we find the following non-zero components

$$\Gamma_{ij}^0 = \frac{\dot{a}(t)}{a(t)} g_{ij} , \qquad (3)$$

$$\Gamma^{i}_{0j} = \frac{\dot{a}(t)}{a(t)} \delta^{i}_{j}, \qquad (4)$$

$$\Gamma^i_{kj} = \frac{Kx^i g_{kl}}{a(t)^2} \,. \tag{5}$$

From these we can calculate the Riemann curvature tensor

$$R^{\mu}_{\ \nu\rho\sigma} = \partial_{\rho}\Gamma^{\mu}_{\nu\sigma} - \partial_{\sigma}\Gamma^{\mu}_{\nu\rho} + \Gamma^{\mu}_{\alpha\rho}\Gamma^{\alpha}_{\nu\sigma} - \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\alpha}_{\nu\rho}. \tag{6}$$

I will not list all non-zero components here since this is not overly illuminating and we are only interested in the Ricci curvature tensor and the Ricci scalar

$$R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu} \,, \qquad R = g^{\mu\nu} R_{\mu\nu} \,. \tag{7}$$

The components of the Ricci tensor are

$$R_{00} = -3\frac{\ddot{a}(t)}{a(t)}, (8)$$

$$R_{0i} = 0, (9)$$

$$R_{ij} = \frac{\ddot{a}(t)a(t) + 2\dot{a}(t)^2 + 2K}{a(t)^2} g_{ij}, \qquad (10)$$

where as expected the isotropy and homogeneity of our metric leads to the vanishing of the vector  $R_{i0} = 0$  and forces the spacial part to be proportional to the metric  $R_{ij} \propto g_{ij}$ . The Ricci scalar is given by

$$R = \frac{6(a(t)\ddot{a}(t) + \dot{a}(t)^2 + K)}{a(t)^2}.$$
 (11)

We recall from lecture 1 that the energy momentum tensor  $T_{\mu\nu}$  is similarly constraint as the Ricci scalar. It can only contain two independent functions of t and its components are

$$T_{00} = \rho(t), \qquad (12)$$

$$T_{0i} = 0, (13)$$

$$T_{ij} = p(t)g_{ij}. (14)$$

Now we can solve Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \,. \tag{15}$$

First let us look at the (00) component

$$-3\frac{\ddot{a}(t)}{a(t)} + \frac{3(a(t)\ddot{a}(t) + \dot{a}(t)^{2} + K)}{a(t)^{2}} - \Lambda = 8\pi G \rho(t)$$

$$\frac{3(\dot{a}(t)^{2} + K)}{a(t)^{2}} - \Lambda = 8\pi G \rho(t).$$
(16)

Dividing both sides by 3 leads to the first Friedmann equations as given in equation (16) in the Lecture 1 notes

$$\frac{\dot{a}(t)^2 + K}{a(t)^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3} \,\rho(t) \,. \tag{17}$$

The mixed components (0i) all vanish and the pure spacial part takes the form

$$\frac{\ddot{a}(t)a(t) + 2\dot{a}(t)^{2} + 2K}{a(t)^{2}}g_{ij} - \frac{3(a(t)\ddot{a}(t) + \dot{a}(t)^{2} + K)}{a(t)^{2}}g_{ij} + \Lambda g_{ij} = 8\pi G p(t)g_{ij} 
\left(-2\frac{\ddot{a}(t)}{a(t)} - \frac{\dot{a}(t)^{2} + K}{a(t)^{2}} + \Lambda\right)g_{ij} = 8\pi G p(t)g_{ij}. (18)$$

Since the metric  $g_{ij} \neq 0$  we can drop it and plug in (17) to get

$$-2\frac{\ddot{a}(t)}{a(t)} - \frac{8\pi G}{3}\rho(t) - \frac{\Lambda}{3} + \Lambda = 8\pi G p(t)$$
 (19)

$$-2\frac{\ddot{a}(t)}{a(t)} + \frac{2}{3}\Lambda = 8\pi G p(t) + \frac{8\pi G}{3} \rho(t).$$
 (20)

Dividing by -2 leads to equation (17) in the lecture notes and the second Friedmann equation

$$\frac{\ddot{a}(t)}{a(t)} - \frac{1}{3}\Lambda = -\frac{4\pi G}{3} \left(\rho(t) + 3p(t)\right) . \tag{21}$$