Cosmology and particle physics

Lecture notes

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Lecture 1 The expanding universe

In this lecture we will learn about the discovery of the expansion of our universe, as well as the fact that the universe is homogeneous and isotropic on scales larger than a few Mpc (megaparsec). This allows us to derive a set of simple equations, the so called Friedmann equations, from general relativity. These equations play a central role in describing the evolution of our universe.

1 The 'Hubble' expansion

When Einstein first wrote down his theory of general relativity in 1915 he was convinced (like most other people) that our universe is static, i.e. the universe as a whole doesn't change in time. However, in 1929 Hubble was able to determine the distances and relative velocities of other galaxies by observing Cepheid variables which led him to a very different universe. In order to understand this, it is useful to first review distance measurements in astrophysics.

1.1 Astronomical unit and parsec

There are a variety of different units used in cosmology and astrophysics. One standard unit in astrophysics is the average distance between the earth and the sun which by definition is one astronomical unit $1au \approx 150 \times 10^6 km = 1.5 \times 10^{11} m$. In cosmology we are interested in larger scales and will mostly use the parsec (pc). The definition of the parsec involves the apparent parallax motion of near stars that is due to the earth's motion around the sun, see figure 1.

From simple trigonometry we find

$$1pc = \frac{1au}{\tan(1'')} \approx \frac{1au}{1''} = \frac{1au}{\frac{1}{60}\frac{1}{60}\frac{\pi}{180}} \approx 2 \times 10^5 au\,, \tag{1}$$

where we used that $\tan(1'') \approx 1''$. We can check that one parsec is roughly the distance light travels in three years $1pc \approx 3.3\,ly$.

By measuring the parallax angle, astronomer can determine the distance of objects that are not too far away. This leads to interesting discoveries that can then be used to determine the distances of much further objects. In particular by studying nearby so called Cepheid variables, astronomers found that these stars pulsate radiately with a well defined relation between their pulsation period and luminosity L. By knowing this relation and the pulsation period we can therefore obtain the stars luminosity L (the total 'light' emitted by the star).

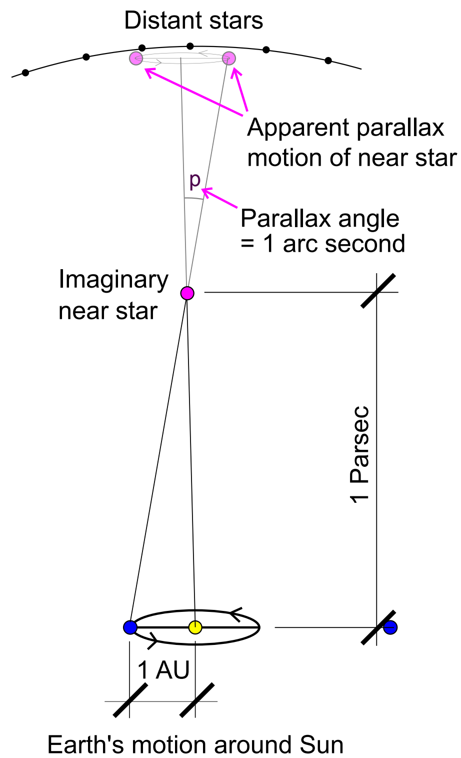

Figure 1: The distance to an imaginary star with a parallax angle of $1'' = 1$ arc second is one parsec (taken from Wikipedia).

The observed flux F then directly gives us the distance of the star since the observed flux decrease with the square of the distance d to the star. In particular we find

$$L = 4\pi\, d^2 F \qquad \Leftrightarrow \qquad d = \left(\frac{L}{4\pi F}\right)^{\frac{1}{2}} . \tag{2}$$

1.2 Hubble's discovery

Hubble studied these Cepheid variables in other galaxies and galaxy clusters and determined their distances using the above equation. In addition he used the Doppler shift of the spectral lines in the star light to determine the relative velocities of these galaxies and galaxy clusters. This lead him to the following plot

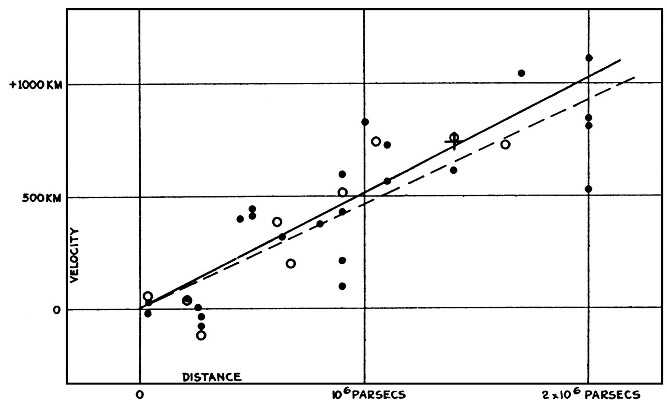

Figure 2: Velocity (in km/s) vs. distance (in parsecs) for galaxies (black dots) and galaxy clusters (circle). The solid line represents a best straight-line fit to the black dots and the dashed line to the circles.

It follows from Hubble's data that the further a galaxy is away from us, the faster it is moving away from us. This observation has been substantially improved over the year, as is shown for example in figure 3.

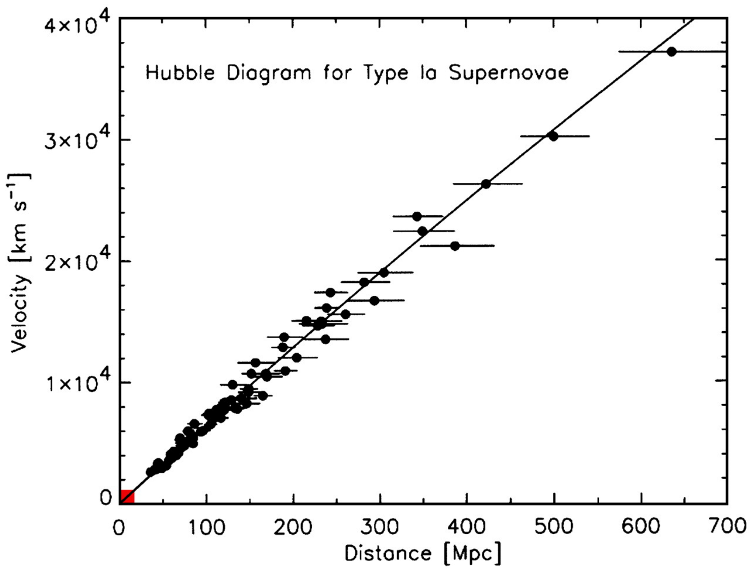

Figure 3: Velocity (in km/s) vs. distance (in Mpc) for Type Ia supernovae (another class of 'standard candles' that allows us to determine distances accurately).

Hubble's original observation is inconsistent with a static universe and instead requires us to consider a universe in which space itself is expanding. This is depicted in figure 4.

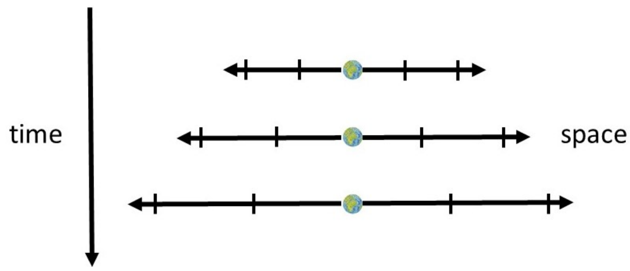

Figure 4: A one dimensional universe in which space itself is expanding. This leads to a linear relation between the relative distance and relative velocity of the earth and any other object in this universe.

Note however, that this does not make the earth or us special in anyway. Any other point in space will observe exactly the same 'Hubble' expansion of the universe, as is shown in figure 5.

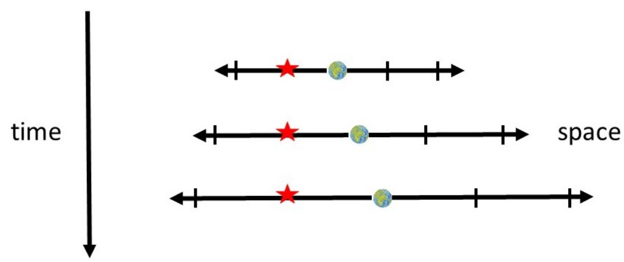

Figure 5: A one dimensional universe in which space itself is expanding. Any point in space will observe the same effect: distant objects are moving away with a velocity that is proportional to their distance.

Before we discuss the equations that describe such a universe, we discuss one more observational fact about our universe in the next subsection.

2 Isotropy and Homogeneity

Trying to describe the time evolution of the entire universe seems like a formidable task and one might wonder how this can be possibly done? In cosmology we are not interested in the details of the evolution on small scales like for example our solar system, but we would like to describe the origin, evolution and the ultimate fate of our universe. But even that seems intractable. Imagine a universe whose evolution is controlled by matter, i.e. at large scales by the evolution of the galaxies. This seems correspond to an N-body problem with N of the order of a few hundred billion ($N \sim 10^{11}$)!

Fortunately, our universe seems highly symmetric at scales larger than a few Mpc. Concretely, there is ample evidence that our universe looks the same in every direction, i.e. it is isotropic, and there are some indications that different locations in the universe allow for the same observation that we make, i.e. the universe is homogeneous. These two properties follow from the so called 'cosmological principle' that postulates that we do not occupy any special place in the (large scale) universe and other observer at any other place in the universe will observe the same properties of the universe.

A simple example of a 3-dimensional isotropic and homogeneous space is the flat space $\mathbb{R}^3$. The line element for this case is

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2 = dx_i^2 \,. \tag{3}$$

Another example of a space that is the same at every point and looks the same in every direction is the 3-sphere S^3 for which we can write the line element as

$$ds^2 = dx_i^2 + dz^2 \,, \qquad x_i^2 + z^2 = a^2 \,. \tag{4}$$

One can prove that the only other such space is given by a hyperspherical surface with negative curvature and line element

$$ds^2 = dx_i^2 - dz^2 \,, \qquad -x_i^2 + z^2 = a^2 \,. \tag{5}$$

By rescaling the x_i and z by a, we can write the last two as

$$ds^2 = a^2 \left(dx_i^2 \pm dz^2 \right) \,, \qquad z^2 \pm x_i^2 = 1 \,. \tag{6}$$

Differentiating $z^2 \pm x_i^2 = 1$ leads to $zdz = \mp x_i dx_i$ and the line element

$$ds^2 = a^2 \left(dx_i^2 \pm \frac{z^2 dz^2}{z^2} \right) = a^2 \left(dx_i^2 \pm \frac{(x_i dx_i)^2}{1 \mp x_i^2} \right) \,. \tag{7}$$

Finally, we introduce the number $K \in \{-1, 0, 1\}$ and combine the three line elements (3), (4), (5) into one single equation

$$ds^2 = a^2 \left(dx_i^2 + K \frac{(x_i dx_i)^2}{1 - K x_i^2} \right) \,. \tag{8}$$

Since we have chosen K to be dimensionless we have to choose the x_i to be dimensionless as well due to the denominator $1 - K x_i^2$. Then the prefactor a needs to have the dimension of a length. In the above equation $K = 0$ corresponds to the flat space case and $K = \pm 1$ to the spherical and hyperspherical case. These three maximally symmetric three dimensional spaces can be similarly defined in two space dimension in which case we can picture them by embedding them into a three dimensional space, as is shown in figure 6.

As can be seen from the figure, the three spaces can be distinguish by measuring the three angels inside a triangle. For example for a sphere one can start at the north pole with a $90°$ angle. These two sides each meet the equator at $90°$ angles and by choosing the third side to lie on the equator, we have constructed a triangle with total interior angles that add up to $270°$! Generically one finds that for spherical geometries the three angles inside a triangle are larger than $180°$, while for hyperspherical spaces they are smaller than $180°$.

Now that we have understood the spacial part of our universe, we can extend the line element to include also time and write (note that we will set the speed of light c equal to 1 so that $1s \approx 3 \times 10^8 m$)

$$ds^2 = -dt^2 + a(t)^2 \left(dx_i^2 + K \frac{(x_i dx_i)^2}{1 - K x_i^2} \right) \,. \tag{9}$$

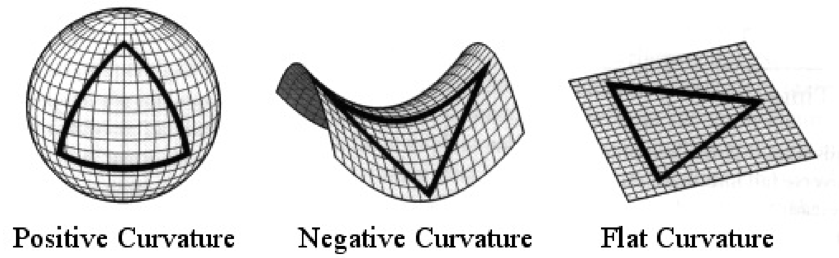

Figure 6: The three possible geometries of our universe.

This is the so called Friedmann-Roberston-Walker metric that is used to describe our universe. Note, that in addition to adding the time coordinate t, we have also allowed the scale factor $a(t)$ to change with time. This scale factor is the function that determines the evolution of our universe. In order to make this more transparent let us first go to spherical polar coordinates

$$dx_i^2 = dr^2 + r^2 d\Omega^2 \,, \qquad d\Omega^2 = d\theta^2 + \sin(\theta)^2 d\phi^2 \,, \qquad x_i dx_i = r dr \,, \tag{10}$$

so that the metric becomes

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - Kr^2} + r^2 d\Omega^2 \right) \,. \tag{11}$$

Now we calculate the distance between an observer at the origin and an object at co-moving radial coordinate r (we always take $a(t) > 0$)

$$d(r,t) = a(t) \int_0^r \frac{dr'}{\sqrt{1 - Kr'^2}} = a(t) \times \begin{cases} \text{arcsin}(r) & K = +1 \\ \text{arcsinh}(r) & K = -1 \\ r & K = 0 \end{cases} \,. \tag{12}$$

This implies that any object at a fixed r moves away from us, if the scale factor $a(t)$ increases with time. More concretely, by differentiating the above equation we can establish the linear relationship between the distance and the velocity

$$v = \frac{\partial d(r,t)}{\partial t} = \frac{\dot{a}(t)}{a(t)} d(r,t) \equiv H \, d(r,t) \,. \tag{13}$$

In the last equation we defined

$$H(t) = \frac{\dot{a}(t)}{a(t)} \,, \tag{14}$$

where $H(t)$ is the so called Hubble parameter since Hubble discovered this linear relationship. So we see that our FRW metric is correctly capturing Hubble's original observation in figure 2 provided that $\dot{a}(t) > 0$.

3 The Friedmann equations

The evolution of the scale factor $a(t)$ is determined by the matter and energy content of the universe using general relativity. If you are not familiar with general relativity then the set of equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}\,, \tag{15}$$

might look rather intimidating and this paragraph might seems rather complicated. In this case you can jump ahead to the next paragraph. In an isotropic and homogeneous universe Einstein's equations boils down to two rather simple equations for $a(t)$. The left-hand-side of the above equation is entirely determined by the FRW metric given above and the cosmological constant Λ. The right-hand-side is determined by the energy-momentum tensor $T_{\mu\nu}$ that encodes the matter and energy in our universe. In a homogeneous and isotropic universe its spacial part has to be proportional to the metric $T_{ij} = p(t)g_{ij}$, where we allowed for an arbitrary time dependent function $p(t)$. The time component $T_{tt} = \rho(t)$ is also an arbitrary function. Finally, the mixed space-time components are a 3-vector. However, such a vector, if non-vanishing, would single out a particular direction which is inconsistent with isotropy that demands that the universe is the same in all directions so we have $T_{ti} = 0$.

Solving Einstein's equations above leads to the following two equations[1] that are called Friedmann's equations

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 + \frac{K}{a(t)^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3}\rho(t)\,, \tag{16}$$

$$\frac{\ddot{a}(t)}{a(t)} - \frac{\Lambda}{3} = -\frac{4\pi G}{3}(\rho(t) + 3p(t))\,. \tag{17}$$

These equations involve three new quantities that deserve further discussion: The parameter Λ is called a cosmological constant and as we will see shortly we can remove it from the equations by shifting ρ and p. So this means that in an homogeneous and isotropic universe we can describe any kind of matter, radiation and energy with just two quantities. What are these and how do we understand them intuitively? A homogeneous universe obviously requires a distribution of energy and matter that does not depend on the spacial coordinates, so instead of dealing with for example empty space dotted with galaxies we can take a continuum limit and think of it as a continuous distribution of matter. You might be familiar with similar approximations when describing air or water. Instead of describing all individual molecules, we describe the whole system as a continuous fluid. The quantity $\rho(t)$ describes the energy density (recall that mass equals energy due to $E = mc^2$) and the function $p(t)$ describes the pressure of this fluid.

By looking at the equations (16), (17) we note that ρ and p can describe a cosmological constant. In particular, if we shift them such that

$$\rho \to \rho - \frac{\Lambda}{8\pi G}\,, \qquad p \to p + \frac{\Lambda}{8\pi G}\,, \tag{18}$$

[1] Depending on your level of familiarity with general relativity I encourage you to either derive these equations yourself or to take a look at the file on the website that gives the detailed derivation.

then we remove Λ and find the Friedmann equations

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 + \frac{K}{a(t)^2} = \frac{8\pi G}{3}\rho(t)\,, \tag{19}$$

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}(\rho(t) + 3p(t))\,. \tag{20}$$

These rather simple equations govern our universe from a split second after the big bang until today. All we need for this is the knowledge of $\rho(t)$ and $p(t)$, i.e. of the matter and energy content of our universe. As we will see in the next lecture, these functions are not too complicated and usually at each time there is one form of energy that is dominating the expressions so that we can solve the Friedmann equations analytically.

Differentiating (19) we get

$$\frac{8\pi G}{3}\dot{\rho}(t) = 2\frac{\dot{a}(t)}{a(t)}\left(\frac{\ddot{a}(t)}{a(t)} - \left(\frac{\dot{a}(t)}{a(t)}\right)^2 - \frac{K}{a(t)^2}\right)\,. \tag{21}$$

Using now equation (19) and (20) and recalling that $H = \dot{a}/a$ we find the *continuity equation*:

$$\dot{\rho}(t) + 3H\left(\rho(t) + p(t)\right) = 0\,. \tag{22}$$

This equation will be useful, when we discuss the different matter and energy content of the universe in the next lecture.