Cosmology and particle physics

Lecture notes

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Lecture 2 Dynamics of the universe

In the last lecture we learned that our universe is homogeneous and isotropic and can be described by the so called FRW metric. Using general relativity one can derive the Friedmann equations that describe the evolution of the universe for any given energy and matter content with an energy density given by $\rho(t)$ and a pressure $p(t)$:

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 + \frac{K}{a(t)^2} = \frac{8\pi G}{3}\rho(t)\,, \tag{1}$$

$$\frac{\ddot{a}(t)}{a(t)} = -\frac{4\pi G}{3}(\rho(t) + 3p(t))\,. \tag{2}$$

From the two Friedmann equations we also derived the continuity equation

$$\dot{\rho}(t) + 3H(t)\,(\rho(t) + p(t)) = 0\,, \tag{3}$$

where the Hubble parameter is defined as

$$H(t) = \dot{a}(t)/a(t)\,. \tag{4}$$

1 The different forms of matter

There are three different forms of matter and energy in our universe and they all satisfy the relation $p(t) = w\rho(t)$, where the constant w is called the equation of state parameter. Plugging this into the continuity equation (3) we can derive the following

$$\begin{aligned}
0 &= \dot{\rho}(t) + 3\frac{\dot{a}(t)}{a(t)}(1+w)\rho(t) \\
0 &= \frac{\dot{\rho}(t)}{\rho(t)} + 3(1+w)\frac{\dot{a}(t)}{a(t)} \\
0 &= \frac{d}{dt}\ln\rho(t) + 3(1+w)\frac{d}{dt}\ln a(t) \\
0 &= \ln(\rho(t)) + 3(1+w)\ln(a(t)) + const. \\
0 &= \ln(\rho(t)) + \ln\left(a(t)^{3(1+w)}\right) + const. \\
1 &= \rho(t)\cdot a(t)^{3(1+w)}\cdot e^{const.} \\
&\Rightarrow \quad \rho(t) \propto a(t)^{-3(1+w)}\,.
\end{aligned} \tag{5}$$

Thus we see that the function $\rho(t)$ and therefore $p(t) = w\rho(t)$ are related to $a(t)$ in a rather simple way. This means, as we will see below, that as long as a single matter component is dominating, we can solve the equations that determine the evolution of our entire universe analytically!

Before we do that let us discuss what kind of matter and energy we expect to have in our universe and derive the corresponding equation of state parameter w.

- **Non-relativistic matter**

  The matter we are most familiar with are stars and galaxies that we can observe at night in the sky. This form of matter has a velocity that is much smaller than the speed of light so that we can neglect its kinetic energy. In a given box in which each side has the initial length $a(t_{in})l$, we have a certain number of stars and galaxies with a mass M. The energy density is then given by $\rho = E/(a(t_{in})l)^3 = M/(a(t_{in})l)^3$, where we used $E = M$ in units where $c = 1$. Now when the universe evolves, the box will change its volume to $a(t)^3 l^3$ as is shown in figure 1.

  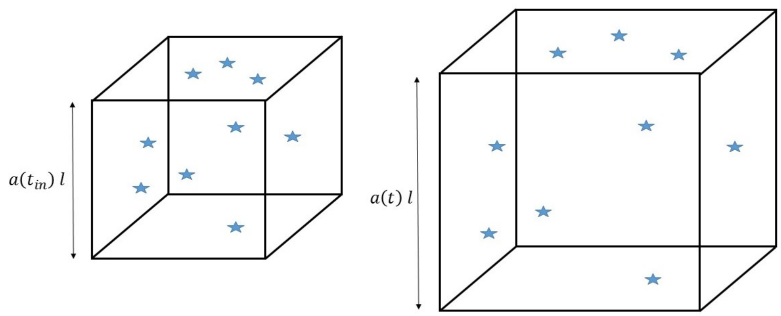

  Figure 1: Non-relativistic matter in an expanding universe.

  Since the mass M stays the same we find the following scaling

  $$\rho_m(t) \propto a(t)^{-3} \quad \Leftrightarrow \quad w = 0 \quad \Leftrightarrow \quad p_m(t) = 0\,. \tag{6}$$

  So we see that non-relativistic matter has an equation of state parameter $w = 0$ and therefore vanishing pressure, which makes sense since the matter inside our box should not exert any pressure on the walls.

  As we will discuss below the largest fraction of cold matter is in the form of an unknown so called *dark matter*.

- **Radiation**

  Another form of energy in the universe is radiation (like for example light). The energy of light in units where $c = \hbar = 1$ is given by $E = 2\pi/(a(t_{in})\lambda)$, where $a(t_{in})\lambda$ is the wavelength. If we have a certain number of photons inside a big volume of initial size $(a(t_{in})l)^3$, then the energy density gets again diluted due to the increase in the volume of the box as above. Additionally due to the expansion of the space the initial wavelength $a(t_{in})\lambda$ increase to $a(t)\lambda$, as shown in figure 2,

  so that we find for radiation

  $$\rho_r(t) \propto a(t)^{-4} \quad \Leftrightarrow \quad w = \frac{1}{3} \quad \Leftrightarrow \quad p_r(t) = \frac{1}{3}\rho_r(t)\,. \tag{7}$$

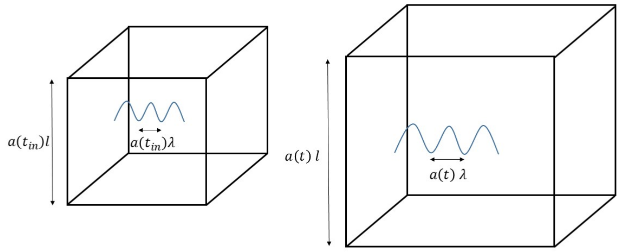

Figure 2: Radiation in an expanding universe.

As we will learn later, our universe is filled with the cosmic microwave background, which is thermal radiation left over from the big bang. Its spectrum is the best measured black body in nature.

- **The cosmological constant**

  As we have seen in lecture 1 in equations (16), (17) and (18), we can describe a cosmological constant by

$$\rho_\Lambda(t) = -p_\Lambda(t) = \frac{\Lambda}{8\pi G} \quad \Leftrightarrow \quad w = -1\,. \tag{8}$$

  So in this case $\rho_\Lambda = -p_\Lambda$ is constant and the energy density does not change in time. This can be understood as follows: During the expansion of the universe more of the vacuum is created and this vacuum has a non-zero energy density ρ so that ρ does not change during the expansion (or contraction).

If you are not too familiar with the theory of general relativity, then you might wonder about the conservation of energy in the above examples. This is a generic feature of general relativity. The would be conservation of energy is replaced by the condition that $\nabla_\mu T^{\mu\nu} = 0$, where ∇_μ denotes the covariant derivative. In particular that means that $\partial_\mu T^{\mu\nu} + \Gamma^\nu_{\mu\sigma} T^{\mu\sigma} + \Gamma^\mu_{\mu\sigma} T^{\sigma\nu} = 0$. Using the handout that derives Friedmann's equations you can check that the above four equations ($\nu = 0, 1, 2, 3$) reduce to the continuity equation (3) for $\nu = 0$ and are trivial otherwise. If you are confused about how this non-conservation of energy is possible in a physical theory, recall that the conservation of energy follows via Noether's theorem from the time-translational symmetry. So any physical theory that is not invariant under time translations can and generically will violate the conservation of energy. An expanding universe is certainly not invariant under time translations so it does violate the standard conservation of energy but it does satisfy the continuity equation that was implied by the two Friedmann equations.

2 The dust filled universe

In this section we will study the simple case of a universe which contains non-relativistic matter, so we set $p(t) = 0$ and we have $\rho(t) \propto a(t)^{-3} > 0$ from (6). The second Friedmann equation (2) then immediately tells us that such a universe cannot be static, i.e. $\ddot{a}(t) \neq 0$.

In fact it tells us that the expansion of the universe is decelerating. This is very intuitive since we know that gravity always attracts. In a universe filled with matter the gravitational attraction between the matter will slow down any initial expansion. There then seem to be three possibilities:

1. The universe will keep expanding for ever at a slower and slower rate.

2. The expansion of the universe will eventually come to a stop.

3. The expansion will slow down and then gravitational attraction between the matter forces the universe to contract and eventually collapse.

We will see that these cases correspond to $K = -1, 0, 1$. We can write equation (1) as

$$0 \leq \dot{a}(t)^2 = \frac{8\pi G}{3}\rho_m(t)a(t)^2 - K = \frac{c_m}{a(t)} - K \,, \tag{9}$$

where we introduced the constant $c_m > 0$ via $8\pi G\,\rho_m(t)/3 = c_m/a(t)^3$ and used (6). We immediately see that for $K = -1$ the right-hand-side can never vanish so in this universe any initial expansion $\dot{a}(t)$ will go on forever. For the case $K = 0$ the right-hand-side vanishes for $a(t) \to \infty$ so the expansion will eventually come to a stop. Finally, for $K = +1$ the first term dominates for very small $a(t)$ but once $a(t) = c_m/K$ the expansion will come to a stop and the universe will then contract (since (1) implies $\ddot{a}(t) < 0$). These three scenarios are shown in figure 3.

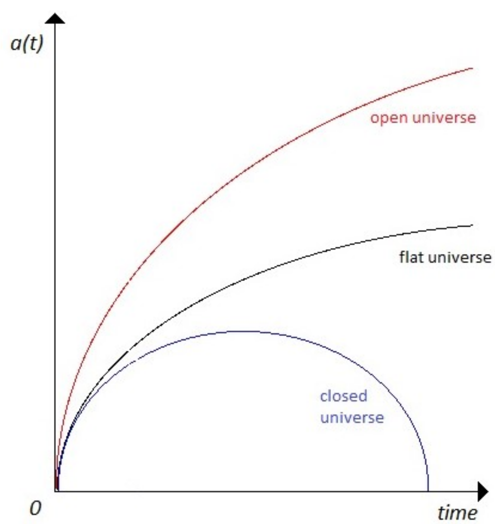

Figure 3: An open, flat and closed universe corresponding to $K = -1, 0, 1$.

We can also rewrite the first Friedmann equation (9) as

$$\frac{1}{2}\dot{a}(t)^2 + V(a(t)) = -\frac{K}{2} \,, \tag{10}$$

with $V(a(t)) = -c_m/(2a(t))$. The above equation describes the motion of a 1-dimensional particle in the potential V and with a total energy $E = -K/2$. Since $V(a(t)) < 0$ we can conclude that for $E \geq 0$, i.e. $K = 0$ and $K = -1$ there exist unbound solutions and the universe can expand forever. For $K = 1$ we have $E = -1/2$ and the trajectories are bound. This is shown in figure 4.

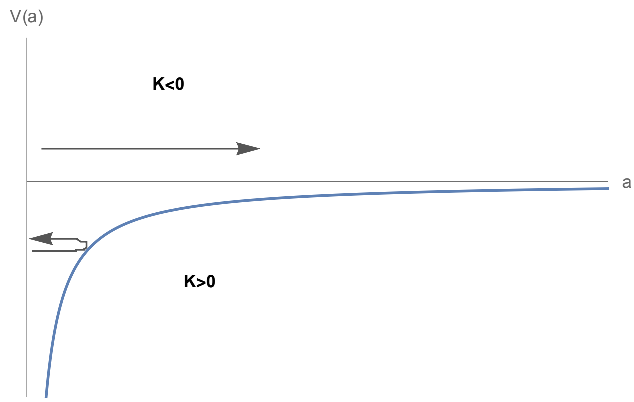

Figure 4: Unbound trajectories only exist for $K = -1$ and $K = 0$, while for $K = 1$ the universe expands and then contracts again.

2.1 A static universe?

Note, that independent of the value of K we find that equation (2) does not permit static solutions for a universe filled with matter. Before the discovery that our universe was expanding, this fact was very troublesome for people like Einstein that imagined our universe to be time independent. Let us therefore try to construct a static universe with matter by adding in the cosmological constant Λ. The universe can then be described by $\rho(t) = \rho_m(t) + \rho_\Lambda$, $p = p_\Lambda = -\rho_\Lambda$. In a static universe with $\dot{a}(t) = \ddot{a}(t) = 0$, we then find from (2) that

$$0 = \rho + 3p = \rho_m - 2\rho_\Lambda \quad \Leftrightarrow \quad \rho_m = 2\rho_\Lambda > 0\,, \tag{11}$$

since $\rho_m > 0$. Using this in (1) gives

$$\frac{K}{a^2} = \frac{8\pi G}{3}\rho = \frac{8\pi G}{3}(\rho_m + \rho_\Lambda) = 8\pi G \rho_\Lambda = \Lambda > 0\,. \tag{12}$$

So we have succeeded in finding a static solution provided that $K = 1$ and $\Lambda > 0$. In this static solution we have $a = 1/\sqrt{\Lambda}$. The important question to ask is whether such a solution is stable. To answer that, we can look again at equation (10). The potential now has an extra contribution from the cosmological constant so that we find for $K = 1$

$$\frac{1}{2}\dot{a}(t)^2 + V(a(t)) = \frac{1}{2}\dot{a}(t)^2 - \frac{c_m}{2\,a(t)} - \frac{1}{6}\Lambda\, a(t)^2 = -\frac{1}{2}\,. \tag{13}$$

A plot of the potential is shown in figure 5.

We see that our static universe corresponds to a maximum of the potential. This means this static universe is unstable. If the matter and energy content is the tiniest bit different, then this universe will either expand forever or collapse. Fortunately our universe is not static, so that we don't have to worry about such delicate solutions.

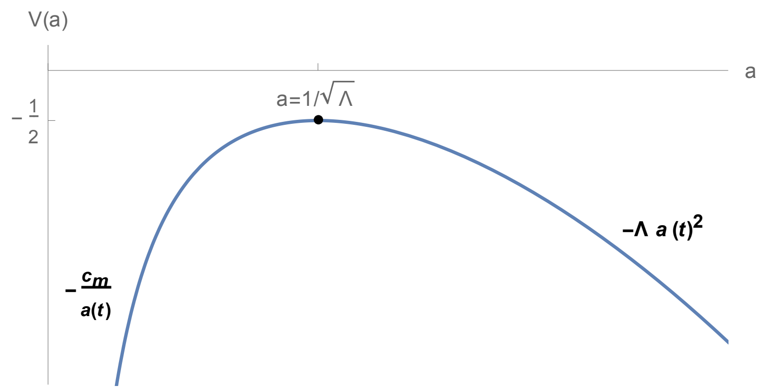

Figure 5: The potential for a static universe with $K = +1$, matter and a cosmological constant $\Lambda > 0$. We see that the static solution with $a = 1/\sqrt{\Lambda}$ is a maximum and therefore unstable.

2.2 The age of the universe

Since we know that our universe is expanding, let us ask the simple question of how old our universe would be, if all its energy would be contained in non-relativistic matter. This is not that bad of an approximation and will give us an age that is of the correct order of magnitude. Before we start the calculation let us introduce an important convention. We call our current time t_0, i.e. we use a subscript 0 to denote today's value of the time variable. Likewise we use $a_0 = a(t_0)$ and $H_0 = H(t_0) = \dot{a}(t_0)/a(t_0)$ to denote today's value of the scale factor and Hubble parameter. Since H_0 by definition is a constant, it is usually called the Hubble constant. There are ever improving measurements of the Hubble constant but its uncertainty is still somewhat large. For that reason one usually writes

$$H_0 = 100h\,\frac{km}{s\,Mpc}\,, \tag{14}$$

where the current experimental value of h is

$$h = .678 \pm .009\,. \tag{15}$$

Hubble's original observations led him to $h \approx 5$ due to several systematic errors. So over the last century astrophysicists reduced the error from a few hundred percent to just a few percent.

Now let us use this value of H_0 to determine the age of a universe filled with non-relativistic matter. We will set $K = 0$ which, as we will discuss below, is very much consistent with observation. We then find from (6) that

$$\rho(t) = \rho_0 \left(\frac{a_0}{a(t)}\right)^3\,, \tag{16}$$

where ρ_0 is the current energy density of the universe. Now we use this in the Friedmann equation (1)

$$a(t)\dot{a}(t)^2 \quad = \quad \frac{8\pi G}{3}\rho_0 a_0^3$$

$$\begin{aligned} \sqrt{a(t)}\;\dot{a}(t) &= \sqrt{\frac{8\pi G}{3}\rho_0 a_0^3} \\ \sqrt{a(t)}\;da &= \sqrt{\frac{8\pi G}{3}\rho_0 a_0^3}\;dt\,, \end{aligned} \tag{17}$$

where in the last line we used $\dot{a}(t) = da/dt$. Now we can integrate both sides which leads to

$$\frac{2}{3}a(t)^{\frac{3}{2}} = \sqrt{\frac{8\pi G}{3}\rho_0 a_0^3}\;t + const. \tag{18}$$

By demanding that $a(t=0)=0$ is the initial singularity we find that the integration constant vanishes. The above equation implies

$$a(t) = a_0 \left(\frac{t}{t_0}\right)^{2/3}\,, \tag{19}$$

since by definition $a(t_0) = a_0$ and t_0 is implicitly defined in (18) but we will not need this particular expression. We rather calculate H_0 directly

$$H_0 = \frac{\dot{a}(t_0)}{a(t_0)} = \frac{\frac{2}{3}a_0\,t_0^{-\frac{1}{3}}/t_0^{\frac{2}{3}}}{a_0} = \frac{2}{3}\frac{1}{t_0}\,. \tag{20}$$

So we have found the age of a matter filled universe in terms of the Hubble constant today

$$t_0 = \frac{2}{3}\frac{1}{H_0} = \frac{2}{3}\frac{s\,Mpc}{100h\,km} \approx \frac{2}{300h}3\times 10^{19}\;s \approx 9.6\times 10^9\;yr = 9.6\;Gyr\,. \tag{21}$$

While this is pretty close to the age of our universe which is roughly 13.8×10^9 years, it is inconsistent with the observation of the oldest stars that are as old as 13×10^9 years.

3 Time evolution of the universe

As we have seen above, a matter dominate universe gives us the right order of magnitude for the age of the universe but the answer is inconsistent with observations. The reason for that is that our universe contains other forms of energy as well. At different points of time different forms of energy density dominate the evolution of the universe. Let us therefore also determine the time dependence of the scale factor for the other cases. We start with the general expression (5) which is equal to

$$\rho(t) = \rho_0 \left(\frac{a(t)}{a_0}\right)^{-3(1+w)}\,. \tag{22}$$

Using this in the Friedmann equation (1) and assuming a negligible curvature contribution ($K=0$), we can repeat the above calculation

$$\left(\frac{\dot{a}(t)}{a(t)}\right)^2 = \frac{8\pi G}{3}\rho_0\left(\frac{a(t)}{a_0}\right)^{-3(1+w)}$$

$$
\begin{aligned}
a(t)^{\frac{1+3w}{2}}\dot{a}(t) &= \sqrt{\frac{8\pi G}{3}\rho_0 a_0^{3(1+w)}} \\
a(t)^{\frac{1+3w}{2}}da &= \sqrt{\frac{8\pi G}{3}\rho_0 a_0^{3(1+w)}}\, dt \\
\frac{2}{3(1+w)}a^{\frac{3(1+w)}{2}} &= \sqrt{\frac{8\pi G}{3}\rho_0 a_0^{3(1+w)}}\, t + const.
\end{aligned} \tag{23}
$$

We can again set the constant to zero by choosing $a(t=0)=0$ and fix the factor of proportionality by demanding that $a(t_0)=a_0$ and get

$$
a(t) = a_0\left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}}\,, \quad w \neq -1\,. \tag{24}
$$

The above derivation doesn't apply to the case of a cosmological constant but in that case one has simply $\rho(t) = const.$ and finds from (1) that

$$
a(t) = a_0 e^{H_0(t-t_0)}\,. \tag{25}
$$

Note that in this case the 'beginning' of the universe is not at $t=0$ but rather at $t=-\infty$. So such a universe is infinitely old. This case is also special since the Hubble parameter $H(t)$ is actually constant (since ρ is constant), while in all other cases it changes with time as

$$
H(t) = \frac{2}{3(1+w)\,t}\,, \quad w \neq -1\,. \tag{26}
$$

Our derivation above also applies to the case of a negatively curved universe with $K=-1$ and $\rho(t)=0$, since this can be thought of as a fluid with energy density $\rho \propto a(t)^{-2}$ which is equal to $w=-1/3$. Let us summarize the different scalings we found

$$
\begin{aligned}
a(t) = &\; a_0\left(\tfrac{t}{t_0}\right)^{\frac{1}{2}}\,, && \text{for a radiation dominated universe, i.e. } w=\frac{1}{3}, && (27)\\
a(t) = &\; a_0\left(\tfrac{t}{t_0}\right)^{\frac{2}{3}}\,, && \text{for a matter dominated universe, i.e. } w=0, && (28)\\
a(t) = &\; a_0\,\tfrac{t}{t_0}\,, && \text{for a curvature dominated universe with } K=-1\text{, i.e. } w=-\frac{1}{3}, && (29)\\
a(t) = &\; a_0 e^{H_0(t-t_0)}\,, && \text{for a universe dominated by } \Lambda\text{, i.e. } w=-1. && (30)
\end{aligned}
$$

It is also interesting to look at the time dependence of the $\rho(t)$ (and therefore also of $p(t) = w\rho(t)$). From equation (26) we find that

$$
\rho(t) = \frac{3}{8\pi G}H(t)^2 = \frac{3}{8\pi G}\left(\frac{2}{3(1+w)}\right)^2\frac{1}{t^2}\,, \quad w \neq -1\,. \tag{31}
$$

This means that in an expanding universe the energy density is diluted as t^{-2} independently of which kind of fluid dominates the energy density.

4 Fun fact

Let us conclude with one non-trivial observation. A universe with a flat geometry $K = 0$ is special in the sense that it can lead to critical evolution (see figure 3) but it also is special since our universe seems to have a very small (or even vanishing) curvature $|K/a_0^2| \ll (\dot{a}(t_0)/a_0)^2$. We have seen in the first lecture that the spatial part of such a universe could be simple the flat space $\mathbb{R}^3$. This is however not the entire truth. It is also possible that one, two or all of the three x_i directions are periodic, i.e. they are circles. This would mean, if these circles wouldn't be too large, we could see ourselves in the sky or we could see the same galaxy twice in the universe by looking in opposite directions. However, up to date there is no evidence of such a non-trivial topology so if the spacial part of our universe is finite (or periodic in any one direction), then the corresponding radius has to be very large and we might never be able to observe this. However, it is interesting to know that our universe (or more precisely a universe with $K = 0$) does not necessarily have to be spatially infinite.