Cosmology and particle physics

Lecture notes

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Lecture 6 The thermal universe - part II

Last time we have seen that the standard model particles in the early universe were interacting so much that the Hubble expansion is negligible compared to the interaction rate, while the temperature is in the range $100GeV \ll T \ll 10^{16}GeV$. This means that there is ample time for the standard model particles to be in thermal equilibrium by the time the temperature is a few hundred GeV. We can therefore use equilibrium thermodynamics to discuss the evolution of this soup of standard model particles as the universe expands and cools.

1 Equilibrium Thermodynamics

In order to understand the number density n, energy density ρ and pressure P [1] for different particles in the early universe we need to know their distribution as a function of phase space, i.e. their distribution in real space and momentum space. For a homogeneous distribution, this phase space function cannot depend on the spacial coordinate $\vec{r}$ and for an isotropic distribution the phase space function can only depend on the absolute value of the momentum $p = |\vec{p}|$. For a system of particles in equilibrium the distribution function is given by the Fermi-Dirac distribution for fermions and by the Bose-Einstein distribution for bosons. Both can be written as

$$f_{\pm}(p) = \frac{1}{e^{(E(p)-\mu)/T} \pm 1}\,, \tag{1}$$

where the $+$ sign is for fermions and the $-$ sign is for bosons and μ denotes the chemical potential.

Due to the ample interactions in the early universe all particles have the same average kinetic energy, i.e. the same temperature T so that we do not need to keep track of different temperatures. In the early universe the chemical potentials for all the particles are small so that we can neglect them and set $\mu = 0$. However, this would mean that the number of particles and anti-particles is the same which isn't quite true as discussed in lecture 5. A small non-zero chemical potential allows one to account for the small matter-anti-matter asymmetry (c.f. the discussion of Bayogenesis above) but it substantially complicates the analysis and is not needed for our discussion.

Allowing for g internal degrees of freedom, i.e. for particles with spin, the particle density in phase space is given by $\frac{g}{(2\pi)^3} f(p)$, where we dropped the subscript $\pm$ to avoid cluttering.

[1] We will switch conventions and denote the pressure by P to avoid confusion with the absolute value of the momentum $p = |\vec{p}|$.

In order to obtain the number density n, we need to integrate this over the momentum

$$n \equiv \frac{g}{(2\pi)^3} \int d^3p f(p) \,. \tag{2}$$

To obtain the energy density ρ, we need to weigh each state by its energy $E(p) = \sqrt{m^2 + p^2}$ so that we have [2]

$$\rho \equiv \frac{g}{(2\pi)^3} \int d^3p f(p) E(p) \,. \tag{3}$$

Lastly the pressure is defined as

$$P \equiv \frac{g}{(2\pi)^3} \int d^3p f(p) \frac{p^2}{3E(p)} \,. \tag{4}$$

The integrals in n, ρ and P have to be evaluated numerically unless we are in particular limits. For such limits we will make use of the general formulas

$$\int_0^\infty du \frac{u^n}{e^u - 1} = \zeta(n+1)\Gamma(n+1) \,, \tag{5}$$

$$\int_0^\infty du u^n e^{-u^2} = \frac{1}{2}\Gamma\left(\tfrac{1}{2}(n+1)\right) \,, \tag{6}$$

where ζ is the Riemann zeta-function and the Γ-function is an extension of the factorial function and in particular takes the values $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}^*$.

1.1 The relativistic limit

Let us first evaluate n, ρ and P for relativistic particles:

$$E(p) = \sqrt{m^2 + p^2} \approx p \gg m \,. \tag{7}$$

We define $y = p/T$ so that $f_\pm(y) = 1/(e^y \pm 1)$. For bosons we then find

$$n_b = \frac{g}{(2\pi)^3} \int_0^\infty \frac{4\pi T^3 y^2 dy}{e^y - 1} = \frac{gT^3 \zeta(3)\Gamma(3)}{2\pi^2} = \frac{\zeta(3)}{\pi^2} g\, T^3 \,, \tag{8}$$

where $\zeta(3) \approx 1.2$. For fermions we can use that

$$\frac{1}{e^y + 1} = \frac{1}{e^y - 1} - \frac{2}{e^{2y} - 1} \,, \tag{9}$$

to get

$$n_f = n_b - \frac{g}{(2\pi)^3} \int_0^\infty \frac{8\pi T^3 y^2 dy}{e^{2y} - 1} = n_b - \frac{g}{(2\pi)^3} \int_0^\infty \frac{\pi T^3 \tilde{y}^2 d\tilde{y}}{e^{\tilde{y}} - 1} = n_b - \frac{1}{4} n_b = \frac{3\zeta(3)}{4\pi^2} g\, T^3 \,. \tag{10}$$

[2] For strongly interacting particles we would have to take into account the interaction energy, but the particles in the early universe were weakly interacting so that we can neglect the interaction energy.

So we have found for relativistic particles that

$$n_b = \frac{4}{3} n_f = \frac{\zeta(3)}{\pi^2} g\, T^3 \,. \tag{11}$$

Note, that the scaling of T^3 agrees with our scaling assumption in equation (5) in lecture 5.

Now let us likewise calculate the energy density

$$\rho_b = \frac{g}{(2\pi)^3} \int_0^\infty \frac{4\pi T^4 y^3 dy}{e^y - 1} = \frac{g}{2\pi^2} T^4 \zeta(4)\Gamma(4) = \frac{\pi^2}{30} g T^4 \,, \tag{12}$$

where we used that $\zeta(4) = \pi^4/90$. For fermions we find

$$\rho_f = \rho_b - \frac{g}{(2\pi)^3} \int_0^\infty \frac{8\pi T^4 y^3 dy}{e^{2y} - 1} = \rho_b - \frac{g}{(2\pi)^3} \int_0^\infty \frac{1}{2} \frac{\pi T^4 \tilde{y}^3 d\tilde{y}}{e^{\tilde{y}} - 1} = \rho_b - \frac{1}{8}\rho_b = \frac{7}{8}\frac{\pi^2}{30} g T^4 \,. \tag{13}$$

So we have

$$\rho_b = \frac{8}{7}\rho_f = \frac{\pi^2}{30} g T^4 \,, \tag{14}$$

where the scaling with the temperature again agrees with the simple dimensional analysis we performed in the last lecture.

Finally, for the pressure P we note that in the relativistic limit $p^2/E(p) = p = E(p)$, so that it trivially follows from the definitions in equations (3) and (4) that for bosons as well as fermions

$$P = \frac{1}{3}\rho = w\rho \,, \tag{15}$$

which nicely agrees with the equation of state parameter $w = \frac{1}{3}$ for radiation.

1.2 Non-relativistic particles

We can also analytically solve for n, ρ and P in the non-relativistic limit, i.e. for regular matter. In this case we have

$$E(p) = \sqrt{m^2 + p^2} \approx m + \frac{p^2}{2m} \,. \tag{16}$$

Let us define $x = p/\sqrt{2mT}$. Since the temperature is related to the average kinetic energy $T \sim \frac{p_{av}^2}{2m}$, which is much smaller than m, we find that $e^{E/T} \approx e^{m/T} \gg 1$. This means that the distribution function, as given in (1), is the same for bosons and fermions

$$f(p) = \frac{1}{e^{E(p)/T} \pm 1} \approx e^{-\frac{E}{T}} \approx e^{-\frac{m}{T}} e^{-x^2} \,. \tag{17}$$

This then gives for the number density

$$n = \frac{g}{(2\pi)^3} e^{-\frac{m}{T}} \int_0^\infty 4\pi (2mT)^{\frac{3}{2}} x^2 e^{-x^2} dx = \frac{g e^{-\frac{m}{T}} (2mT)^{\frac{3}{2}} \Gamma\left(\frac{3}{2}\right)}{4\pi^2} = g\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{m}{T}} \,, \tag{18}$$

where we used that $\Gamma(3/2) = \sqrt{\pi}/2$.

In order to calculate the energy density, we use that $E \approx m$ and find to leading order from the definition in equation (3) that

$$\rho = mn = gm \left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{m}{T}} \,. \tag{19}$$

Finally, we again calculate the pressure P as given in (4). Here we use that $p^2/E \approx p^2/m = (2mT)x^2/m$ and find, using the simplification leading to (18), that

$$P = \frac{g}{(2\pi)^3}\frac{e^{-\frac{m}{T}}}{3m}\int_0^\infty 4\pi(2mT)^{\frac{5}{2}}x^4 e^{-x^2}dx = \frac{ge^{-\frac{m}{T}}(2mT)^{\frac{5}{2}}\Gamma\left(\frac{5}{2}\right)}{12\pi^2 m} = gT\left(\frac{mT}{2\pi}\right)^{\frac{3}{2}} e^{-\frac{m}{T}} = nT \,, \tag{20}$$

where we used that $\Gamma(5/2) = 3\sqrt{\pi}/4$. Note that this is the familiar ideal gas law $P = nk_BT$ or after multiplying by the volume V: $PV = Nk_BT$.

Now as we argued above, the temperature T is much smaller than the mass m. This means that

$$P = nT = \frac{T}{m}\rho = w\rho \approx 0 \quad \text{for} \quad T \ll m \,. \tag{21}$$

So we again reproduce our previous result that for non-relativistic matter we can neglect the pressure.

The exponential decay of the number density for non-relativistic particles that we found in (18) can be understood as particle-anti-particle annihilation. As mentioned in lecture 5, at high energies particle-anti-particle pairs are created and destroyed at equal rates but once the universe cools so much that the mass of a particle starts to become important, the pair creation stops and the annihilation leads to an exponential suppression in the number density of the respective particle.

2 The effective number of relativistic species

The total radiation density is given by the sum over the contributions from all particles

$$\rho_r = \sum_i \rho_i = \frac{\pi^2}{30} g_\star(T) T^4 \,, \tag{22}$$

where i runs over all standard model particles and $g_\star(T)$ is the effective number of degrees of freedom at temperature T, which we take to be the photon temperature. The sum over i can receive two contributions. One from relativistic particles that are in equilibrium with the photons, i.e. that have $T_i = T \gg m_i$. These contribute to $g_\star$ as follows

$$g_\star^{th}(T) = \sum_{i=bosons} g_i + \frac{7}{8}\sum_{i=fermions} g_i = g_b + \frac{7}{8} g_f \,, \tag{23}$$

where th stands for thermal equilibrium. However, particles can decouple so that they won't be in thermal equilibrium with the photons anymore. If these particles are relativistic, i.e. we have $T_i \neq T$ and $T_i \gg m_i$, then they contribute to $g_\star$

$$g_\star^{dec}(T) = \sum_{i=bosons} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_{i=fermions} g_i \left(\frac{T_i}{T}\right)^4 \,, \tag{24}$$

where *dec* stands for decoupled. We thus have

$$g_\star(T) = g_\star^{th}(T) + g_\star^{dec}(T) \,. \tag{25}$$

As we discussed last time, at $T \gg 100GeV$ all standard model particles are relativistic (cf. figure 3) and in thermal equilibrium with the photons (and each other).

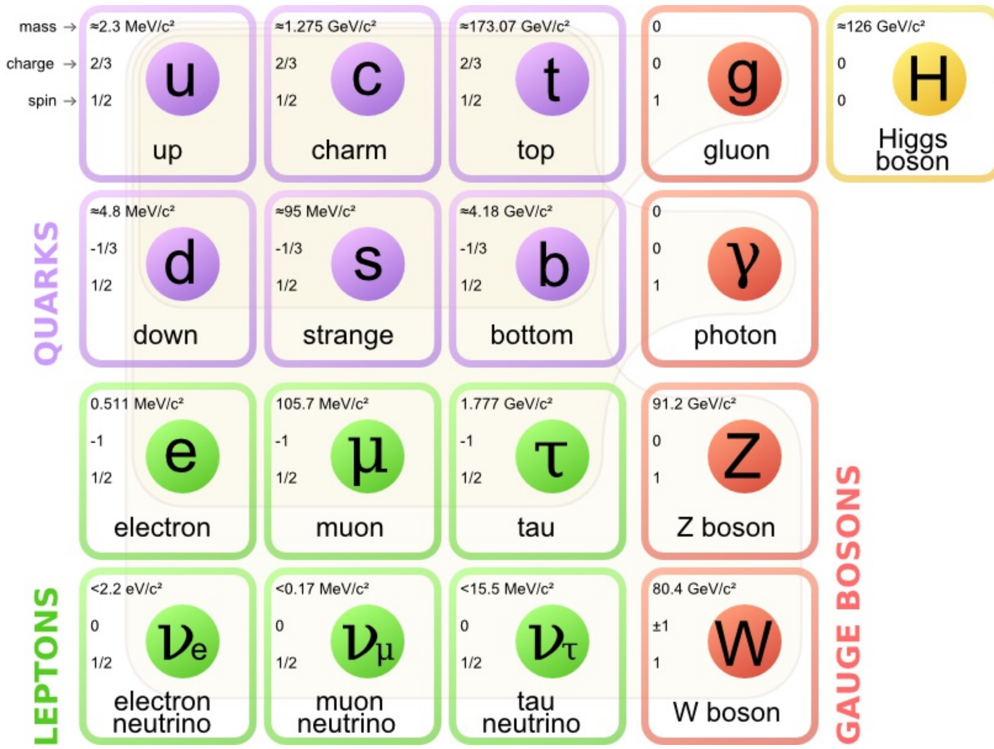

Figure 1: The known particles in our universe (taken from Wikipedia).

So let us calculate $g_\star$ for the standard model. We have the following contribution to g_b:

- The Z, W^+, W^- and the photon γ are all massless vectors and have two degrees of freedom each. Therefore they contribute $4 * 2 = 8$.
- Before the electroweak phase transition the Higgs scalar is a two vector whose entries are complex scalars so that it contributes $2 * 2 = 4$.
- There are actually 8 gluons[3] that are all massless vectors, so that they contribute $8 * 2 = 16$.

This leads to a total of $g_b = 28$.

Massive fermions have two possible spins and therefore have two internal degrees of freedom each. We take the neutrinos to be only left-handed, i.e. we assume that the right-handed neutrinos are very heavy and do not contribute. Fermions also have antiparticles that we need to include in our counting. Then we find the following contributions to g_f:

[3]This is not clear from figure 3 and follows from the more complicated nature of the strong force. The gluons are the entries in an $SU(3)$ matrix that has eight real independent entries.

- The left-handed neutrinos and their anti-particles contribute $3 * 2 = 6$.
- The electron e, the μ and the τ contribute twice as much $3 * 2 * 2 = 12$.
- The six quarks can have three distinct charges under the strong force[4] which leads to an additional factor of 3 so that we have $6 * 2 * 2 * 3 = 72$.

Thus in total we have $g_f = 90$ and the value of $g_\star$ at temperatures well above a $100GeV$ is

$$g_\star = g_b + \frac{7}{8} g_f = 28 + \frac{7}{8} 90 = 106.75\,. \tag{26}$$

In an expanding and cooling universe particles will become non-relativistic. Before we discuss this in detail in the next subsection, let us mention the electroweak phase transition: At a temperature around $100GeV$ the standard model of particle physics undergoes a transition during which the Higgs field develops a vacuum expectation value. This vacuum expectation value is actually what gives a mass to all the fields (particles) in the standard model. After this phase transition the $W^\pm$ and Z gauge fields have a mass. Massive vectors have three internal degrees of freedom so that this modifies our counting above. However, these new three degrees of freedom (one for each W^+, W^- and Z) come from the Higgs field that after this transition is only left with a single degree of freedom. So the net number of degrees of freedom does not change during the electroweak phase transition.

3 Particle freeze-out

Once the temperature of the universe drops below the mass of a particle, the particle-anti-particle annihilation for this particle is favored compared to particle-anti-particle creation. This leads to an exponential decay of the particle number n, as derived in equation (18). This transition from relativistic to non-relativistic particle and the resulting annihilation of particles with their anti-particles is not instantaneous. Roughly 80% of the particles are annihilated in the interval $m > T > m/6$.

One effect of this so called particle freeze out is, as we will discuss next time, that the decrease in temperature of the universe is slowed down, since the particle-anti-particle annihilation deposits the energy contained in the annihilating particles into the remaining particles that are still in thermal equilibrium. But what happens to the particles themselves? Do they completely disappear?

In a non-expanding (but still somehow cooling) universe with vanishing chemical potential for these particles, this would be the case and the number density would keep decreasing exponentially with the temperature. However, as we mentioned last time (cf. equation (12) in the lecture 5 notes), in our universe there is a small baryon-anti-baryon asymmetry so that the particles cannot all annihilate since there aren't enough anti-particles around. This leads to the observed remaining baryons in our universe. Note, that in the standard model of particle physics heavy quarks (and leptons) decay to the lighter quarks (and leptons). For example, the top quark has an estimated lifetime of $5 \times 10^{-25}s$ so that any relic top quarks will quickly decay to up quarks.

[4] Each of them is a three vector on which the gluon $SU(3)$ matrix can act.

Another fate of non-relativistic particles in an expanding universe is that at a certain point their interaction rate Γ (which is proportional to their exponentially decaying number density n) becomes so small that it is smaller than the Hubble expansion H. In such a case the particles and anti-particles cannot find each other anymore and the annihilation stops. The exponential decay in the particle density followed by this so called freeze-out is shown in the log-log-plot in figure 2.

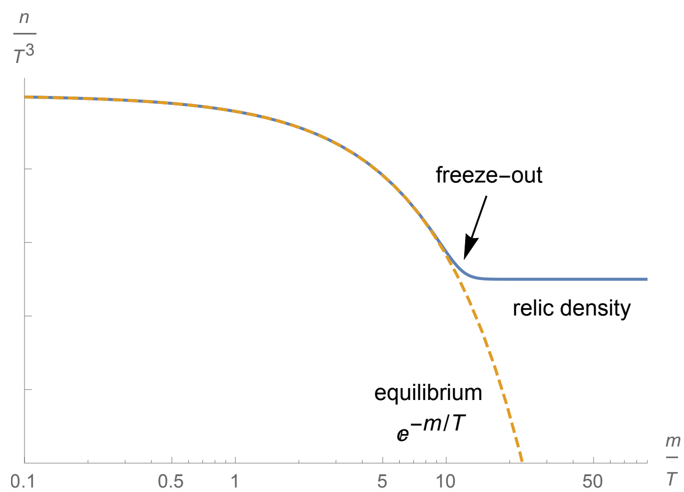

Figure 2: Once a particle becomes non-relativistic its number density decays exponentially. The Hubble expansion or a non-zero chemical potential can lead to a relic density.

4 Evolution of the relativistic degrees of freedom

Having briefly discussed the potential fate of relativistic matter that becomes non-relativistic, let us return to the relativistic degrees of freedom of the standard model, during the time when our universe cools from a few hundred GeV to a few eV. The behavior of $g_\star(T)$ is shown in figure 3.

After the electroweak phase transition particles have their usual mass and the heaviest field, the top quark starts to become non-relativistic. This reduces $g_\star$ by $12 * 7/8 = 10.5$. Next the massive vector bosons $W^\pm$ and Z and the Higgs scalar become non-relativistic, which reduces $g_\star$ by another $9 + 1 = 10$. After that the b and c quarks and the τ become also non-relativistic. At a temperature of roughly $150MeV$ our universe undergoes another phase transition. The strong force becomes so strong that all quarks and gluons combine into uncharged bound states. For example, the u and d quarks combine into protons uud and neutrons udd. As we discussed last time, all the particles that are combinations of three quarks are called baryons and even the lightest of them, the proton, has a mass of $1GeV \gg 150MeV$ so that after the QCD phase transition the baryons are all non-relativistic. However, there are also bound states of quarks and anti-quarks, the so so called mesons. The lightest mesons are the pi-mesons $\pi^+ = u\bar{d}$, $\pi^- = d\bar{u}$ and the π^0 which is a combination of $u\bar{u}$ and $d\bar{d}$. These three mesons have a mass of $135MeV$-$140MeV$ so that they will still be relativistic after the QCD transition. They become non-relativistic shortly before the μ

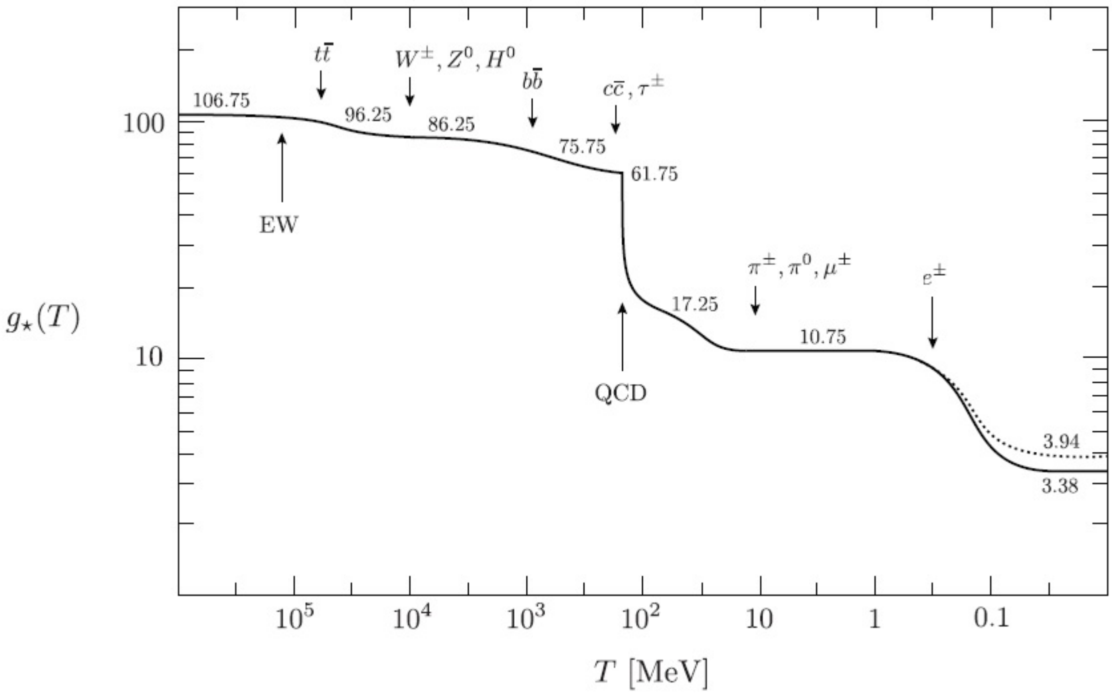

Figure 3: The evolution of the relativistic degrees of freedom in our early universe (taken from Daniel Baumann's "Cosmology" lectures).

leaving only the electron e, the photons and neutrinos as relativistic particles. As you can see from the graph something interesting is happening once the electrons become non-relativistic and we will discuss this next time.