Deriving the energy momentum tensor for a scalar field

The energy momentum tensor is defined as the variation of the action with respect to the metric $g_{\mu\nu}$. For inflation we are interested in the action of a scalar field that is given by

$$S = \int d^4x\sqrt{-g}\mathcal{L} = \int d^4x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right) . \tag{1}$$

Before we vary this action with respect to the metric $g_{\mu\nu}$ we recall the variations of $\sqrt{-g} = \sqrt{-\det(g_{\mu\nu})}$ and the inverse metric $g^{\mu\nu}$:

$$\begin{aligned} \delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}}\delta g = \frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\delta g_{\mu\nu}\,, & (2)\\ \delta g^{\mu\nu} &= -g^{\mu\alpha}g^{\nu\beta}\delta g_{\alpha\beta}\,, & (3) \end{aligned}$$

where in the first line we used Jacobi's formula $\delta g = g\,g^{\mu\nu}\delta g_{\mu\nu}$.

Now we can calculate the energy momentum tensor for a single scalar field

$$\begin{aligned} T^{\mu\nu} &= \frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g_{\mu\nu}} \\ &= \frac{2}{\sqrt{-g}}\left(\frac{1}{2}\sqrt{-g}\,g^{\mu\nu}\mathcal{L} + \sqrt{-g}\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}}\right) \\ &= g^{\mu\nu}\left(-\frac{1}{2}g^{\alpha\beta}\partial_\alpha\phi\,\partial_\beta\phi - V(\phi)\right) + g^{\mu\alpha}g^{\nu\beta}\partial_\alpha\phi\,\partial_\beta\phi \\ &= g^{\mu\nu}\left(-\frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi - V(\phi)\right) + \partial^\mu\phi\,\partial^\nu\phi\,. \end{aligned} \tag{4}$$

Lowering the indices we find

$$\begin{aligned} T_{00} &= \rho_\phi = \frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi + V(\phi) + \dot{\phi}^2\,, \\ T_{ij} &= P_\phi g_{ij} = g_{ij}\left(-\frac{1}{2}\partial_\alpha\phi\,\partial^\alpha\phi - V(\phi)\right) + \partial_i\phi\,\partial_j\phi\,. \end{aligned} \tag{5}$$

Recalling the FRW metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a(t)^2\gamma_{ij}dx^i dx^j \equiv -dt^2 + a(t)^2\left(dx_i^2 + K\frac{x_i^2 dx_i^2}{1-Kx_i^2}\right) , \tag{6}$$

we can read of the energy density and pressure for a scalar field [1]

$$\begin{aligned} \rho_\phi &= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\frac{(\nabla\phi)^2}{a^2} + V(\phi)\,, & (7)\\ P_\phi &= \frac{1}{2}\dot{\phi}^2 - \frac{1}{6}\frac{(\nabla\phi)^2}{a^2} - V(\phi)\,, & (8) \end{aligned}$$

where $(\nabla\phi)^2 = \gamma^{ij}\partial_i\phi\,\partial_j\phi$ with γ^{ij} being the inverse of the γ_{ij} defined in equation (6). This is the expected result and we see that a slowly varying scalar field indeed behaves like a cosmological constant since $\rho_\phi \approx -P_\phi$.

[1] To get P_ϕ we can use $g^{ij}T_{ij} = g^{ij}g_{ij}P_\phi = 3P_\phi$.