PALP & the classification of reflexive polytopes

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@ Extremal Laurent polynomials / Warwick, Oct. 19–21 (2009)

- Motivation: ~~classification~~ → examples (CYs)
- Reflexive polytopes /w H. Skarke ('94 – '00)
  - duality & Newton polytopes
  - IP polytopes and weights
  - lists, normal forms & all that
- PALP /w H. Skarke [math.SC/0204356]
  - poly.x class.x cws.x & nef.x
    - conifolds /w V. Batyrev [arXiv:0802.3376]
    - statistics of lattice polytopes [arXiv:0809.1188]
- PALP++ wishlists

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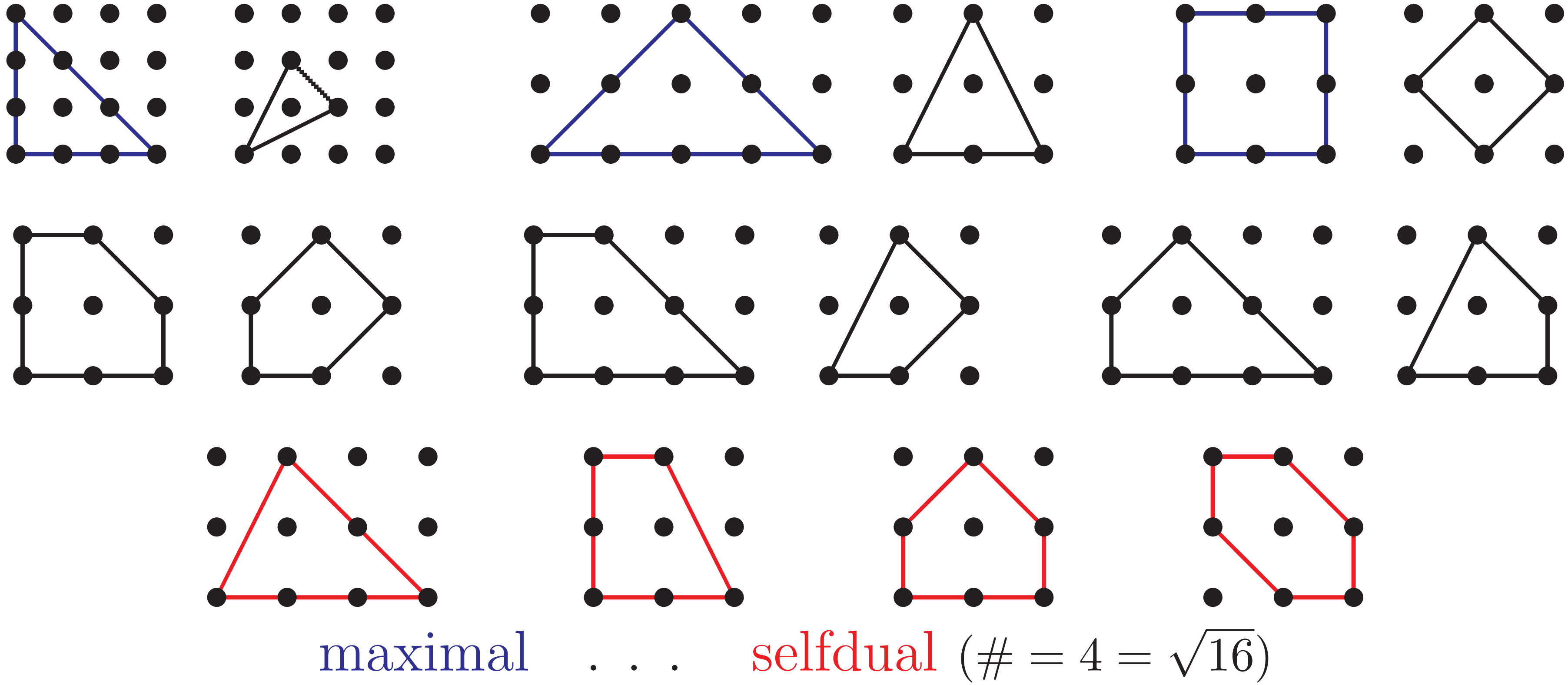
Enumerating reflexive polytopes

- Batyrev 1993: Calabi-Yau condition $\Leftrightarrow$ reflexive polytopes
  - Polar/dual pair $\quad \Delta \subseteq M_{\mathbb{R}} \quad \Delta^\circ \subseteq N_{\mathbb{R}} \;\Leftrightarrow\;$ saturate $\langle \Delta, \Delta^\circ \rangle \geq -1$
  - reflexive pair $\;\Leftrightarrow\;$ both are lattice polytopes

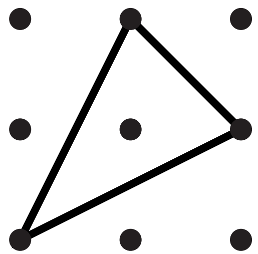


- Δ maximal $\Leftrightarrow$ Newton polytope of quasihomogeneous polynomial
- weights $=$ barycentric coordinates of IP-simplices $\subseteq \;\Delta^\circ \;\leftarrow$ minimal

maximal Δ $\rightarrow$ enumerate all reflexive **subpolytopes** on **sublattices**

$\updownarrow$

minimal Δ°



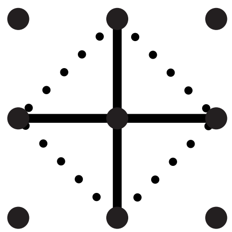
IP-simplex **3** $\Rightarrow$ linear relation $\sum w_j \vec{v}_j = 0$

weight vector $\vec{w}$, degree $D = \sum w_j$

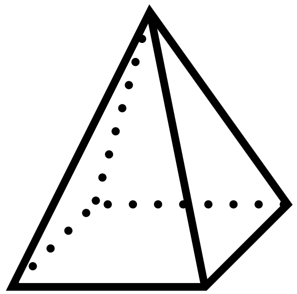
$q_j = w_j / D$ = barycentric coord. of 0 w.r.t. v_i

Def: IP-weight vector $\Leftrightarrow$ Newton polytope $\Delta_D(\vec{w})$ **has interior point**

Non-simplicial minimal Δ° $\Rightarrow$ **weight matrix** w_{ij}



2×2



$\underbrace{3+3}$

$$\begin{matrix} w_{11} & w_{12} & w_{13} & 0 & 0 \\ w_{21} & 0 & 0 & w_{24} & w_{25} \end{matrix} \qquad D_i = \sum_j w_{ij}$$

Lemma: In each dimension there is a finite number of IP weight vectors.
Lines of IP weight matrices are IP weight vectors.

Old def.: w.-system = w.-vector, combined w.-system (CWS) = w.-matrix

IP confined polytopes [arXiv:0809.1188]

- An IP polytope (Calabi-Yau polytope) is a lattice polytope with a unique interior point (choose IP=0)
  Their number is finite for fixed dimension.

- IP simplices $\mapsto$ weight vectors
  Almost one-to-one: take the lattice $\langle v_j \rangle_{\mathbb{Z}})$ with $\sum w_j v_j = 0$

- IP-weight (definition in PALP): $\Delta(\vec{w})$ is IP polytope
  non IP-weights: $\Delta_{20}(1,5,6,8)$ $\Delta_{20}(1,1,5,5,8)$ $\Delta_{23}(2,2,2,3,3,11)$

- Def.: Δ is called IP-confined if $\widetilde{\Delta} := \mathrm{ConvHull}(\Delta^* \cap \mathbb{Z}^d)$ is IP polytope.
  - IP weights are weight vectors of IP confined simplices!
  - $\exists$ very efficient enumeration algorithm! ... selecting points in $\Delta(w)$

- An IP polytope is called IP closed if $\widetilde{\widetilde{\Delta}} = \Delta$.
  - IP-closed $\Rightarrow$ reflexive in ≤ 4 dim. transversal counter-ex.: $\Delta_7(1,1,1,1,1,2)$
  - For IP-closed Δ the involution $\Delta \to \widetilde{\Delta}$ extends reflexive duality

$\ldots w_4$	D	$w_1 \ldots w_4$	D	$w_1 \ldots w_4$	D	$w_1 \ldots w_4$	D	$w_1 \ldots w_4$	D	$w_1 \ldots w_4$	D	$w_1 \ldots w_4$
1 1 1	12	2 2 3 5	16	1 3 4 8	20	1 4 5 10	24	3 4 5 12	30	5 6 8 11	36	3 4 11 18
1 1 2	12	1 2 4 5	16	1 2 5 8	21	3 5 6 7	24	2 3 7 12	30	3 4 10 13	36	1 5 12 18
1 2 2	12	1 2 3 6	17	2 3 5 7	21	1 5 7 8	24	1 3 8 12	30	4 5 6 15	38	5 6 8 19
1 1 3	12	1 1 4 6	18	3 4 5 6	21	2 3 7 9	25	4 5 7 9	30	2 6 7 15	38	3 5 11 19
1 2 3	13	1 3 4 5	18	1 4 6 7	**21**	**1 4 7 9**	26	2 5 6 13	30	1 6 8 15	40	5 7 8 20
2 2 3	14	2 3 4 5	18	2 3 5 8	21	1 3 7 10	26	1 5 7 13	30	2 3 10 15	42	3 4 14 21
1 2 4	14	2 2 3 7	18	2 3 4 9	22	2 4 5 11	26	2 3 8 13	30	1 4 10 15	42	2 5 14 21
2 3 3	14	1 2 4 7	18	1 3 5 9	22	1 4 6 11	27	5 6 7 9	32	4 5 7 16	42	1 6 14 21
1 3 4	15	3 3 4 5	18	1 2 6 9	22	1 3 7 11	27	2 5 9 11	32	2 5 9 16	**44**	**5 8 9 22**
2 3 4	15	2 3 5 5	19	3 4 5 7	24	3 6 7 8	**28**	**3 7 8 10**	**33**	**5 8 9 11**	44	4 5 13 22
2 2 5	15	1 3 5 6	20	2 5 6 7	24	4 5 6 9	28	4 6 7 11	33	3 5 11 14	48	3 5 16 24
1 3 5	15	1 3 4 7	20	3 4 5 8	**24**	**2 5 8 9**	28	3 4 7 14	34	4 6 7 17	50	7 8 10 25
2 3 5	15	1 2 5 7	**20**	**1 5 6 8**	24	1 6 8 9	**28**	**1 5 8 14**	34	3 4 10 17	54	4 5 18 27
3 3 4	16	1 4 5 6	20	2 4 5 9	24	3 4 7 10	28	1 4 9 14	36	7 8 9 12	66	5 6 22 33
3 4 4	16	2 3 4 7	20	2 3 5 10	24	2 3 8 11	**30**	**4 7 9 10**	**36**	**3 7 8 18**		

The 104 IP-simplices in $d = 3$ correspond to 95 transversal and 9 (boldface) non-IP-weights.

Lists and normal forms [arXiv:math.SC/0204356]

- find all reflexive subpolytopes: "keeplist"; drop vertices of "bad facets"

- huge redundancy in subpolytopes! need linear order $\longrightarrow$ bisection search
  Define normal form using (coordinate independent) vertex pairings VPM

  concretely: search 400 million = 4GB every milli-second ($\rightarrow$ compress!)

```
echo "5 1 1 1 1 1 /Z5: 0 1 1 3 0" | poly.x -fm
#GL(Z,4)-Symmetries=8, #VPM-Symmetries=120
5 5  Pairing matrix of vertices and equations of P
   0    5    0    0    0
   0    0    0    5    0
   0    0    5    0    0
   0    0    0    0    5
   5    0    0    0    0
```

Define "intrinsic" basis by minimizing VPM $\forall$ permutations of lines+columns $\longrightarrow$ "upper-triangular" coordinates

Stabilizer = VPM-symmetry; broken by lattice to GLZ-symmetry

```
echo "5 1 1 1 1 1 /Z5: 0 1 1 3 0" | poly.x -fSN
```

→ Symmetry, Normal form

```
#GL(Z,4)-Symmetries=8, #VPM-Symmetries=120
4 5  Normal form of vertices of P    perm=24013
   1    0    0    1   -2
   0    1    1    0   -2
   0    0    5    0   -5
   0    0    0    5   -5
```

Affine normal form: add one dimensions $(1, \Delta)$

```
echo "5 1 1 1 1 1 /Z5: 0 1 1 3 0" | poly.x -fA
4 5
   1    1    0    3    0
   0    5    0    0    0
   0    0    5    0    0
   0    0    0    5    0
```

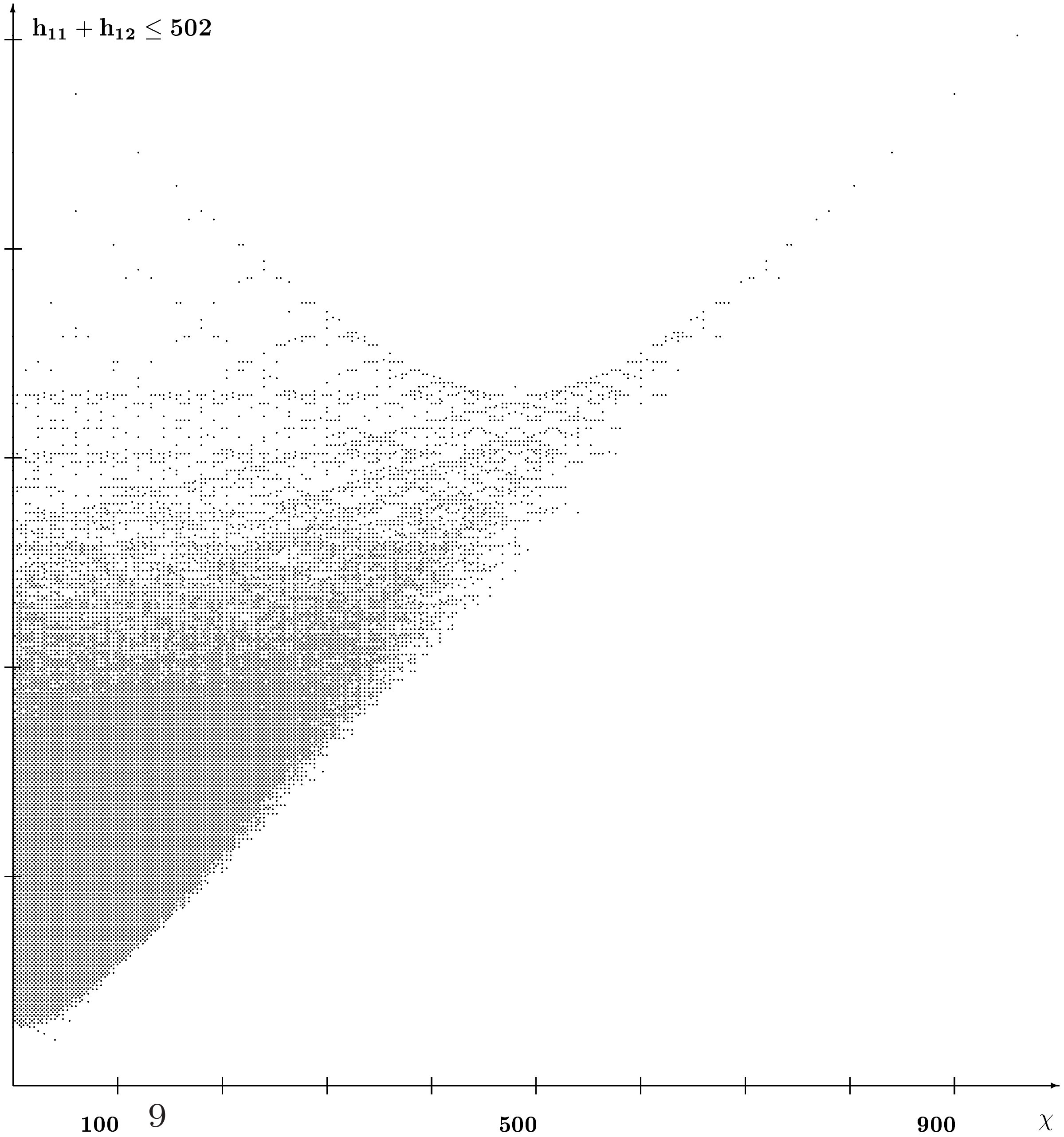
```
echo "5 1 1 1 1 1 /Z5: 0 1 1 3 0" | poly.x -ft
```

t=trace NF-computaion

```
...
Poly NF try[119]:   C=20314
  -1   0   0   2  -1 =>   1   0   3   1  -5
  -1  -1   4  -1  -1 =>   0   1   3   0  -4
  -1  -1  -1   4  -1 =>   0   0   5   0  -5
  -1  -1  -1  -1   4 =>   0   0   0   5  -5
Poly NF:  NormalForm=try[10]  #Sym(VPM)=120  #Sym(Poly)=8
V_perm made by Poly_Sym (order refers to VertNumList):
01234
02134
20431
10432
41230
24031
42130
14032
4 5
   1   1   0   3   0
   0   5   0   0   0
   0   0   5   0   0
   0   0   0   5   0
```

4 dimensions: [hep-th/0002240]

- 184.026 weights, 308+25+7 maximal reflexive polyhedra
- 473.800.776 reflexive polyhedra
- 30.108 pairs of Hodge numbers
- 4.5 GB database
  (internet search mask)
- test: mirror symmetry !



- poly.x -h

```
This is ''poly.x'': computing data of a polytope P
Usage:   poly.x [-<Option-string>] [in-file [out-file]]
Options (concatenate any number of them into <Option-string>):
  h  print this information            | n  do not complete polytope or
  f  use as filter                     |      calculate Hodge numbers
  g  general output:                   | i  incidence information
     P reflexive: numbers of (dual)    | s  check for span property
       points/vertices, Hodge numbers  |      (only if P from CWS)
     P not reflexive: numbers of       | I  check for IP property
       points, vertices, equations     | S  number of symmetries
  p  points of P                       | T  upper triangular form
  v  vertices of P                     | N  normal form
  e  equations of P/vertices of P-dual | t  traced normal form computation
  m  pairing matrix between vertices   | V  IP simplices among vertices of P*
       and equations                   | P  IP simplices among points of P*
  d  points of P-dual                  |      (with 1<=codim<=# when # is set)
       (only if P reflexive)           | Z  lattice quotients for IP simplices
  a  all of the above except h,f       | #  #=1,2,3  fibers spanned by IP
  l  LG-'Hodge numbers' from single    |      simplices with codim<=#
       weight input                    | ## ##=11,22,33,(12,23): all (fibered)
  r  ignore non-reflexive input        |      fibers with specified codim(s)
  D  dual polytope as input (ref only) |    when combined: ### = (##)#
Input:    degrees and weights 'd1 w11 w12 ... d2 w21 w22 ...'
```

- `poly.x -x` x = extended/experimental

```
Test/new options:   A  affine normal form
                    B  Barycenter and lattice volume [# ... points at deg #]
                    F  print all facets
                    G  Gorenstein: divisible by I>1
                    L  like 'l' with Hodge data for twisted sectors
                    U  simplicial facets in N-lattice
                    U1 Fano (simplicial and unimodular)
                    U4 Fanos from reflexive projections (M lattice)
                    U5 ::U4 but don't compute lifts if inFILE==stdin
                    U6 ::U5 but more efficient [2 maximal missing]
                    C1 conifold CY (unimod with square codim 2 faces
                    C2 conifold FANO (divisible by 2 & basic 2 faces
                    z  fatness (4d)
```

Example: search for "divisible polytopes", i.e. Gorenstein index > 1 takes 13 hours

```
class.x -b -di ~/tcy/d4/zzdb -vf 5 -vt 27 | poly.x -fG > OUTput
```

- `class.x -h`

```
This is  'class.x', a program for classifying reflexive polytopes
Usage:     class.x  [options] [ascii-input-file [ascii-output-file]]
Options:   -h           print this information
           -f or -      use as filter; otherwise parameters denote I/O files
           -m*          various types of minimality checks (* ... lvra)
           -p* NAME     specification of a binary I/O file (* ... ioas)
           -d* NAME     specification of a binary I/O database (DB) (* ... ios)
           -r           recover: file=po-file.aux, use same pi-file
           -o[#]        original lattice [omit up to # points] only
           -s*          subpolytopes on various sublattices (* ... vphmbq)
           -k           keep some of the vertices
           -c           check consistency of binary file or DB
           -M[M]        print missing mirrors to ascii-output
           -a[2b]       create binary file from ascii-input
           -b[2a]       ascii-output from binary file or DB
           -H*          applications related to Hodge number DBs (* ...cstfe)
```

- `class.x -x`

```
Extended/experimental options:
           -A[2B]             AffineNF to Binary for non-IP
           -B[2A]             Binary to AffineNF for non-IP
        -sh ... gen by codim>1 points (omit IPs of facets)
        -sp ... gen by all points
        -sb ... generated by dim<=1 (edges), print if rank=2
        -sq ... generated by vertices,       print if rank=3
            q,b currently assume that dim=4
           -d1 -d2 [-po]      combined mirror info (projected
```

• **cws.x -h**

```
This is ‘cws.x’: create weight systems and combined weight systems.
Usage:   cws.x -<options>;     the first option must be ‘w’, ‘c’, ‘i’, or ‘h’.
Options: -h        print this information

         -w# [L H] make IP weight systems for #-dimensional polytopes.
                   For #>4 the lowest and highest degrees L<=H are required.
             -r/-t make reflexive/transversal weight systems (optional).

         -c#       make combined weight systems for #-dimensional polytopes.
                   For #<=4 all relevant combinations are made by default,
                   otherwise the following option is required:
             -n[#] followed by the names wf_1 ... wf_# of weight files
                   currently #=2,3 are implemented.
              [-t] followed by # numbers n_i specifies the CWS-type, i.e.
                   the numbers n_i of weights to be selected from wf_i.
                   Currently all cases with n_i<=2 are implemented.

          -i       compute the polytope data M:p v [F:f] N:p [v] for all IP
                   CWS, where p and v denote the numbers of lattice points
                   and vertices of a dual pair of IP polytopes; an entry
                   F:f and no v for N indicates a non-reflexive ‘dual pair’.
              -f   use as filter; otherwise parameters denote I/O files
```

- `nef.x -h` ... complete intersections

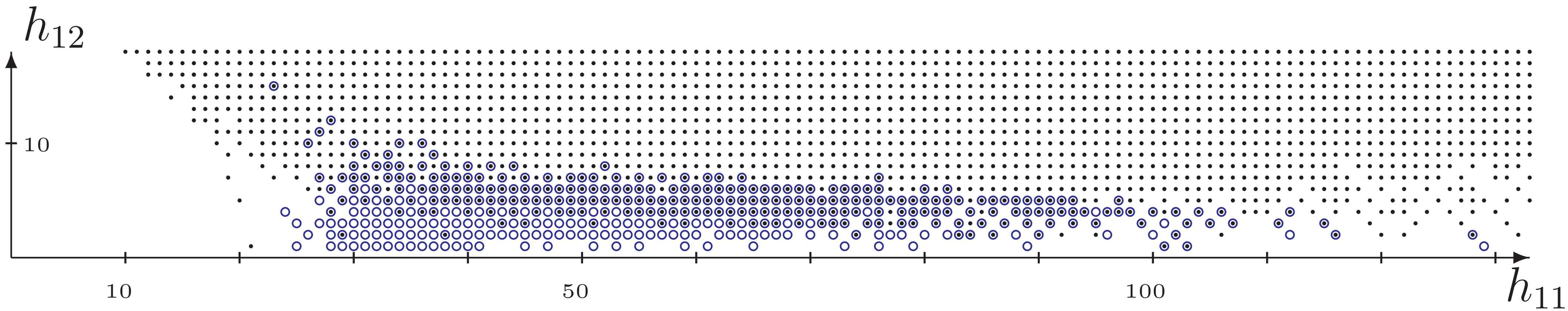
```
Usage:   cws.x -<options>

Options: -h         print this information
         -f or -    use as filter; otherwise parameters denote I/O files
         -N         starting-poly is in N-lattice (default is M)
         -H         gives full list of hodge numbers
         -Lv        prints L vector of Vertices (in N-lattice)
         -Lp        prints L vector of Points (in N-lattice)

         -p         prints only Partitions, no Hodge numbers
         -D         calculates also direct products
         -P         calculates also projections
         -t         full time info

         -cCODIM    codimension (default = 2)

         -Fcodim    FIBRATIONS up to codim (default = 2)
```

- **New CYs from conifold transitions:** w/Batyrev [0802.3376]
  - blow down $\mathbb{P}^1$, flat deformation $0 \mapsto S^3$ $\Rightarrow$ reduce h_{11}
  - Sigularity type: (only): conifold curves in ambient space

    $\Rightarrow$ combinatorial: 2-faces are minimal triangles or squares

    473 800 776 reflexive polytopes (4.5GB) ... 1 day on desktop $\rightarrow$ 198 849

    Smoothable [Namikawa]: 198 849 $\rightarrow$ 30241new CY 3-folds



- Many new CYs with small h_{11}
- Candelas and Davies [arXiv:0809.4681]: quotients $\rightarrow$ small $h_{11} + h_{12}$

$h_{11} = 1$: 8871 CYs with h12 = 21,23-51,53,55,59,61,65,73,76,79,89,101,103,129

210 smooth: h12 = 25,28-41,45,47,51,53,55,59,61,65,73,76,79,89,101,103,129

$h_{11} = 2$: 43080 CYs with h12 = 22,24-80,82-90,96,100,102,103,111,112,116,128

3470 smooth: h12 = 26,28-60,62-68,70,72,74,76,77,78,80,82-84,86,88,90,96,100,102,112,116,128

$h_{11} = 3$: ...

h_{11}	$\#(\Delta)_C$	$\#(\Delta)_H$	$\#(Euler)_C$	$\#(Euler)_H$	#(diffeo. types)
1	210	5	30	5	69
2	3470	36	60	18	??
3	11389	244	68	42	
4	10264	1197	72	87	
5	3808	4990	66	113	
6	815	17101	47	128	
7	140	50376	26	149	
8	35	128165	10	158	
9	3	...			

Picard number $h_{11} = 1$

- Thm. (C.T.C. Wall): diffeomorphism type $\leftrightarrow$ tripple intersections and linear form $c_2 \cdot H_i$ (for torsion-free cohomology)
- 210 polytopes $\rightarrow$ 69 diffeomorphism types with 30 Euler numbers

Mirror proposal

symplectic surgery condition: Smith, Thomas and Yau [math/0209319]
mirror conifold singularities, but: possibly additional singularities

- We computed 30 PF operators (of 109)
  - up to 13 different polytopes / CY
  - up to 5 different principal periods / CY !!!

  Conjecture [hep-th/0410018]: same instanton numbers,
  PF operators related by rational transformations
  (so far verified in all computable cases)

Picard Fuchs operators: $\theta = t\,\frac{d}{dt}$

$$
\begin{aligned}
\theta^4 &+\frac{2}{29}\ t\ \theta(24\theta^3-198\theta^2-128\theta-29)-\frac{4}{841}\ t^2\ (44284\theta^4+172954\theta^3+248589\theta^2+172057\theta+47096)\\
&-\frac{4}{841}\ t^3\ (525708\theta^4+2414772\theta^3+4447643\theta^2+3839049\theta+1275594)\\
&-\frac{8}{841}\ t^4\ (1415624\theta^4+7911004\theta^3+17395449\theta^2+17396359\theta+6496262)\\
&-\frac{16}{841}\ t^5\ (\theta+1)(2152040\theta^3+12186636\theta^2+24179373\theta+16560506)\\
&-\frac{32}{841}\ t^6\ (\theta+1)(\theta+2)(1912256\theta^2+9108540\theta+11349571)\\
&-\frac{10496}{841}\ t^7\ (\theta+1)(\theta+2)(\theta+3)(5671\theta+16301)-\frac{24529152}{841}\ t^8\ (\theta+1)(\theta+2)(\theta+3)(\theta+4)
\end{aligned}
$$

- The 210 polytopes for 1-parameter CYs have **up to 28 vertices!**
- The PF operators are mostly (except for 3) in the list of CY-equations by [G. Almkvist, C. van Enckevort, D. van Straten, W. Zudilin]
- but: at least 1 case with 6th order operator [Almkvist, van Straten]

Statistics of reflexive lattice polytopes

Skarke's formula (empirical):

$$N_d \approx 2^{2^{d+1}-4} \qquad \Rightarrow \qquad N_5 \approx 1.2 \cdot 10^{18} \quad N_6 \approx 2.1 \cdot 10^{37}$$

Statistics of lattice polytopes?

all faces of reflexive polytopes! $\longrightarrow$ "reflexive dimension"

Random generation:

m mirror pairs in p polytopes: $\Rightarrow N = p^2/(2m)$

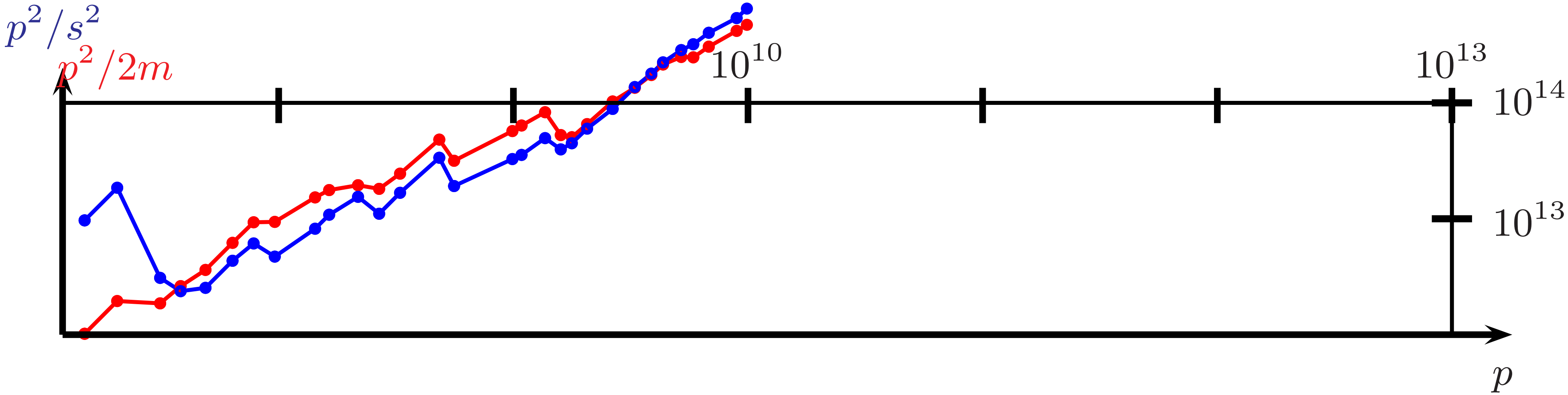
s self-mirror among p polytopes: $\Rightarrow N = (p/s)^2$

Current version of PALP: smallest "r-maximal" polytope has 47 points (≤ 680 in 4d)

```
echo "24 3 3 4 4 10"|class.x -f -po /tmp/zz
100kR-0 1MB 277kIP 110kNF-0k 6_47 v16r15 f26r25 55b21 20s 19u 8n
200kR-0 2MB 563kIP 234kNF-0k 7_34 v17r17 f27r25 55b21 43s 42u 17n
...
800kR-1280 11MB 2461kIP 968kNF-12k 11_46 v17r17 f28r27 85b24 182s 180u 72n ...
```

3 minutes:

```
24 3 3 4 4 10 R=798878 +1280sl hit=0 IP=2461059 NF=968137 (12612)
Writing /tmp/zz: 798878+1280sl 1181m+14s 9MB  pp/2m=2.68615e+08 pp/ss=3.25615e+09
```



Sample: Transversal+Reflexive with $36 \leq p \leq 65$ points.

... run out of disk space

Possible strategies:

- Approximation from below:

  Probably possible in 5d: some 10^9 weight systems

  Skarke's algorithm for weight systems: find M-lattice points

  Limit point number: Skarke's algorithm hard to generalize!

- **CICYs $\subset$ reflexive Gorenstein cones [Batyrev, Borisov]**

  generalized → Cayley Calabi–Yau: divisible reflexive polytopes Stabilization (Candelas et al.'s CICYs 1-parameter at codimension 4; Batyrev: cone construction)

PALP++ wishlist

- we have
  - binary tree and binary compression database infrastructure
  - many special purpose routines
  - historical problem: specify limits (dim, #points, ...) @ compilation
- we'ld like to have
  - fully dynamical dimensions, arbitrary/taylored precision
  - better modularization → more flexible access to basic routines
  - ray representation → non-reflexive polytopes, cones
  - triangulations, intersection rings, PF operators . . . (using Singular)
    - ∗ enumerate diff-classes (Wall's criterion) for small Pic
    - ∗ use toric CICYs as backbone of the web of CYs (Reid's fantasy)
  - synergies with the SAGE project