Consider chiral superfields

$$\begin{aligned}\Phi^i &= x^i - \partial x^i\theta^+\theta^- - \bar\partial x^i\bar\theta^+\bar\theta^- + \partial\bar\partial x^i\theta^+\theta^-\bar\theta^+\bar\theta^- \\ &\quad - \sqrt{2}\left(\psi^i\theta^+ - \bar\partial\psi^i\theta^+\bar\theta^+\bar\theta^- + \tilde\psi^i\bar\theta^+ - \partial\tilde\psi^i\bar\theta^+\theta^+\theta^-\right) + 2F^i\theta^+\bar\theta^+.\end{aligned} \tag{1}$$

and antichiral superfields

$$\begin{aligned}\Phi^{\bar\imath} &= x^{\bar\imath} + \partial x^{\bar\imath}\theta^+\theta^- + \bar\partial x^{\bar\imath}\bar\theta^+\bar\theta^- + \partial\bar\partial x^{\bar\imath}\theta^+\theta^-\bar\theta^+\bar\theta^- \\ &\quad - \sqrt{2}\left(\psi^{\bar\imath}\theta^- + \bar\partial\psi^{\bar\imath}\theta^-\bar\theta^+\bar\theta^- + \tilde\psi^{\bar\imath}\bar\theta^- + \partial\tilde\psi^{\bar\imath}\bar\theta^-\theta^+\theta^-\right) + 2F^{\bar\imath}\theta^-\bar\theta^-.\end{aligned} \tag{2}$$

The first challenge is to show

$$\int -\frac{1}{2}\Phi^i\Phi^{\bar\imath}d^4\theta\, d^2z = \int\left(\bar\partial x^i\partial x^{\bar\imath} + \partial x^i\bar\partial x^{\bar\imath} + 2\psi^i\bar\partial\psi^{\bar\imath} + 2\tilde\psi^i\partial\tilde\psi^{\bar\imath} + 2F^iF^{\bar\imath}\right)\, d^2z. \tag{3}$$

You need to use the fact the total derivatives integrate to zero.

Now for the harder problem. Consider

$$\int K(\Phi^i,\Phi^{\bar\jmath})\, d^4\theta\, d^2z, \tag{4}$$

where K is an arbitrary analytic function of chiral superfields and antichiral superfields.

You want to image expanding K:

$$K = a_0 + a_i\Phi^i + a_{\bar\imath}\Phi^{\bar\imath} + a_{ij}\Phi^i\Phi^j + a_{i\bar\jmath}\Phi^i\Phi^{\bar\jmath} + \ldots \tag{5}$$

So for each term we need to find the 4 θ's within the product of superfields.

Define

$$g_{i\bar\jmath}(x) = \left.\frac{\partial^2K}{\partial\Phi^i\partial\Phi^{\bar\jmath}}\right|_{\theta=0}, \tag{6}$$

Where the subscript means set all the θ's to zero.

The second challenge is to show:

$$\begin{aligned}\int K\, d^4\theta d^2z = -2\int g_{i\bar\jmath}\Big(&\partial x^i\bar\partial x^{\bar\jmath} + \bar\partial x^i\partial x^{\bar\jmath} + 2\psi^i\bar D\psi^{\bar\jmath} + 2\tilde\psi^iD\tilde\psi^{\bar\jmath} + 2F^iF^{\bar\jmath} \\ &- 2g_{i\bar k,j}\psi^i\tilde\psi^jF^{\bar k} - 2g_{\bar\imath k,\bar\jmath}\psi^{\bar\imath}\tilde\psi^{\bar\jmath}F^k - 2g_{i\bar\jmath,k\bar l}\psi^i\psi^{\bar\jmath}\tilde\psi^k\tilde\psi^{\bar l}\Big)\, d^2z,\end{aligned} \tag{7}$$

Here the comma subscript notation means take a derivative:

$$g_{ij,k} = \frac{\partial g_{ij}}{\partial x^k}.$$

The D's are covariant derivatives pulled back to the worldsheet. For example,

$$\bar D\psi^j = \frac{\partial\psi^j}{\partial\bar z} + \Gamma^j_{ik}\frac{\partial x^i}{\partial\bar z}\psi^k,$$

were Γ is the connection given by $g_{i\bar{\imath}}\Gamma^{i}_{jk} = g_{j\bar{\imath},k}$ etc., as if $g_{i\bar{\jmath}}$ were a Kähler metric (which indeed it is!). Note that the Γ's vanish if they have indices mixed between barred and unbarred. E.g.,

$$\Gamma^{k}_{i\bar{\jmath}} = 0.$$

The field F appears without any derivatives and so can be eliminated using the equations of motion:

$$\begin{aligned} F^{i} &= \Gamma^{i}_{jk}\psi^{j}\tilde{\psi}^{k} \\ F^{\bar{\imath}} &= \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}}\psi^{\bar{\jmath}}\tilde{\psi}^{\bar{k}}. \end{aligned} \tag{8}$$

You final challenge is to show that substituting this back into (7) gives the $N = (2,2)$ **supersymmetric nonlinear σ-model**

$$S = \frac{1}{4\pi\alpha'}\int d^{2}z\left(g_{i\bar{\jmath}}\left(\partial x^{i}\bar{\partial}x^{\bar{\jmath}} + \bar{\partial}x^{i}\partial x^{\bar{\jmath}} + 2\psi\bar{D}\psi^{\bar{\jmath}} + 2\tilde{\psi}^{i}D\tilde{\psi}^{\bar{\jmath}}\right) + 2R_{i\bar{\jmath}k\bar{l}}\psi^{i}\psi^{\bar{\jmath}}\tilde{\psi}^{k}\tilde{\psi}^{\bar{l}}\right), \tag{9}$$

where R is the Riemann curvature tensor.